Stabilization of the Negative Mass Instability in a Rotating Relativistic Electron Beam

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It is shown that the negative mass instability in a rotating relativistic electron layer may be stabilized by a radial dc electric field of a suitable magnitude. The stabilization mechanism is independent of the beam velocity spread, and is insensitive to the beam current, the container geometry, or the azimuthal mode number. A simple stability criterion is given.

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The negative mass instability[1] poses a major obstacle to the development of high-current cyclic accelerators. Various methods of stabilization have been proposed and analyzed. Notably, the effects of a moderate beam angular velocity spread and betatron oscillations have been considered.[2-6] For a betatron, the addition of a toroidal magnetic field[7] has been shown to reduce the instability growth rate considerably[5,6] and for the Astron, the proximity of the container walls to the relativistic electron layer (E layer) stabilizes the lower azimuthal modes.[8]

In this Letter, we show that by imposing a negatively biased radial electric field of a suitable strength, the negative mass instability may be suppressed. This stabilization differs from all previously known mechanisms in that it is effective even for a very cold beam; it does not require a toroidal magnetic field, nor is it sensitive to the container geometry, the beam current, or the toroidal mode number. The simple stability criterion, given in Eq. (10) below, does not seem to be very stringent for electron beams in the megaelectronvolt range.

Our finding is based on an analytic treatment of the stability of the E layer situated in a configuration similar to the Astron, which has been shown[8] to include all essential features of the negative mass instability. We limit our study to a highly ordered beam whose unperturbed orbits are concentric circles. Such a beam should yield the most pessimistic prediction as far as the beam stability is concerned; hence our analysis is conservative. The simplicity of the assumed equilibrium orbits allows the linear stability theory to be formulated *exactly*, including *all* ac and dc space-charge effects, *all* relativistic effects, and *all* electromagnetic effects, for general equilibrium profiles. As we shall see, our dispersion relation reproduces the standard results in the appropriate limits. For example, the diocotron instability is recovered, and the negative mass instability removed, in the planar, nonrelativistic limit.

Consider a cylindrical E layer with radial density profile $n_0(r)$ which, in equilibrium, circulates concentrically with azimuthal velocity $\vec{v}_0(r)=\hat{\theta}v_0(r)\equiv\hat{\theta}r\omega_0(r)$ under the combined action of an axial magnetic field $\vec{B}_0=\hat{z}B_0(r)$ and radial electric field $\vec{E}_0=\hat{r}E_0(r)$. These fields include both the self-fields and the externally imposed fields. We assume that the E layer is located between two cylindrical conductors of inner and outer radii a and b, respectively, and that there is no axial motion nor axial variation in either the unperturbed or the perturbed states.

The governing equations for the equilibrium read

$$\gamma_0 v_0^2/r=-(e/m_0)(E_0+v_0B_0), \tag{1}$$

$$dB_0/dr=-\mu_0 J_0\equiv-\mu_0 e n_0 v_0, \tag{2}$$

$$r^{-1}d(rE_0)/dr=n_0 e/\epsilon_0. \tag{3}$$

Here, e and m_0 are respectively the electron charge and rest mass, μ_0 and ϵ_0 are the free-space permeability and permittivity, and $\gamma_0=(1-v_0^2/c^2)^{-1/2}$ is the relativistic mass factor with c being the speed of light. Once the electron density $n_0(r)$, the total electrostatic potential difference between $r=a$ and $r=b$, and the magnetic flux are specified, the unperturbed fields $v_0(r)$, $E_0(r)$, and $B_0(r)$ are to be solved from (1) to (3) to yield a self-consistent equilibrium solution.

We next consider a small-signal perturbation on such an equilibrium. All perturbations are assumed to vary as $f(r)\exp(i\omega t-il\theta)$, where l is the azimuthal mode number and ω is the (complex) eigenfrequency to be determined. In the absence of axial variation and of axial motion, the TM modes and the TE modes are decoupled. The nontrivial components of the rf electromagnetic fields are E_r, E_θ, and B_z for the TE modes. The Maxwell equations, the Lorentz force law, and the continuity equation may then be linearized and combined to yield the following second-order ordinary differential equation for $\phi = rE_\theta$:

$$\frac{d}{dr}\left[\frac{1}{r\rho}\left(1+\frac{N_D}{\gamma_0^2}\right)\frac{d\phi}{dr}\right]+\phi\left[\frac{\omega_0}{\Omega}\frac{\omega^2}{c^2}\frac{d\eta_D}{dr}+\frac{dA}{dr}+B\right]=0. \tag{4}$$

In this equation

$$\Omega = \omega - l\omega_0(r), \quad N_D = \omega_p^2/\omega_0^2 D, \quad \rho = 1 - l^2c^2/\omega^2 r^2 + N_D\Omega^2/\omega^2,$$

$$\omega_p^2 = e^2 n_0/\gamma_0 m_0 \epsilon_0, \quad D = PQ - \Omega^2/\omega_0^2, \quad P = \gamma_0^2(1+h), \quad Q = h + v_0'/\omega_0,$$

$$\eta_D = N_D(\omega_0/\omega)P/\gamma_0^2, \quad A = -\eta_D(l/\rho r^2)(1+N_D\beta_0^2\Omega/l\omega_0),$$

$$\beta_0 = v_0/c, \quad B = (\omega^2/c^2 r)(1-\eta_D^2/\rho + N_D/\gamma_0^2).$$

In the definition of Q, a prime denotes a derivative with respect to r, and h is proportional to the equilibrium electric field and is defined by

$$h = -erE_0/m_0\gamma_0^3 v_0^2. \tag{5}$$

Note that h is positive (negative) if the equilibrium electric field points radially outward (inward). The eigenvalue ω is determined by solving (4) subject to the boundary conditions $\phi = 0$ at $r = a$ and at $r = b$.

Equation (4) is completely general and of wide applicability. It governs the small-signal stability properties of various devices including the Astron,[3,4,8,9] gyrotron,[10] orbitron,[11] and cross-field microwave devices,[12] depending on the parameters of the electron beam as long as the equilibrium states are modeled by Eqs. (1)–(3). A detailed comparative stability study of various types of equilibrium will be given elsewhere. For the present purpose, we restrict ourselves to an E layer with uniform density n_0 extending from $r = r_1$ to $r = r_2$. The E-layer thickness $\tau = r_2 - r_1$ is assumed to be much less than the mean radius R. We shall use τ/R as an expansion parameter. Furthermore, we assume that $|\Omega| << \omega_0$, a condition readily satisfied by the negative mass mode.[1-10]

The instability growth rate ω_i may be analytically derived from Eq. (4) for a thin E layer by expanding about the singularity $\Omega = 0$ in the complex r plane. To two orders in τ/R, it is given by

$$\omega_i^2 = \left(\frac{1}{b_+ + b_-}\right)\omega_p^2\left(\frac{l\tau}{R}\right)\frac{(\beta_0^2+2h)}{(1+\gamma_0^2 h^2)} + \frac{l^2\tau^2}{R^2}\Lambda\omega_0^2, \tag{6}$$

where b_+ (b_-) is the normalized wave admittance at the outer (inner) edge of the E layer,[13] and

$$\Lambda = \frac{1}{4}\frac{1}{\gamma_0^6(1+h)^2(1+\gamma_0^2h^2)} \times\left\{\left(\frac{\omega_p^2}{\omega_0^2}\right)^2[\beta_0^2+2h+(1+h)^2]+2\left(\frac{\omega_p^2}{\omega_0^2}\right)\gamma_0^4(\beta_0^2+2h)^2-\gamma_0^6(1+\gamma_0^2h^2)(\beta_0^2+2h)^2\right\} - \left[\frac{1}{2}\frac{(1+h)\omega_p^2/\omega_0^2}{\gamma_0^2(1+\gamma_0^2h^2)}\left(\frac{b_+-b_-}{b_++b_-}\right)\right]^2. \tag{7}$$

The first-order term (in τ/R) of (6) describes the negative mass effect[1] and dc field effects, while the second-order term includes the diocotron effect[14] and finite-thickness stabilization.[3] The derivation of (6) will be given elsewhere. Its validity may be tested as follows:

(a) If the beam is infinitesimally thin, and if we ignore the dc electric field by setting $h = 0$, we then recover from (6) the well-known dispersion relationship $\omega_i^2 \cong (l\tau/R)\omega_p^2\beta_0^2/(b_+ + b_-)$ for the negative mass instability for the Astron geometry.[8,3,4,10]

(b) A more stringent test on the validity of (6) is to consider the planar geometry limit. In this limit, we let $R \to \infty$, $l \to \infty$, $\omega_0 \to 0$, but require that v_0, $k_y = l/R$, E_0, and τ remain finite. Then $h \to \infty$ from (5) and the first term of the right-hand side of (6) tends to zero, consistent with the notion that there is no negative mass instability in a planar geometry. Using only the last term of (6), we then obtain

$$\omega_i^2 \simeq (k_y^2\tau^2/4\gamma_0^8h^2)\omega_p^4/\omega_0^2 = (k_y^2\tau^2/4\gamma_0^4)\omega_p^4/\omega_c^2$$

which agrees with the well-known growth rate for the diocotron instability for a sheet beam.[14] In writing the last expression, we have used the self-consistent equilibrium condition $E_0 + v_0 \times B_0 = 0$ (for a planar sheet beam) in (5) and defined $\omega_c \equiv |e|B_0/m_0\gamma_0$. This agreement with previously known results adds to our confidence in the dispersion relationship (6), especially with regard to the effects of dc self-fields. Recent work on the diocotron instability is reported by Tsang and Davidson.[15]

For a thin E layer with sufficiently high energy ($\gtrsim 1$ MeV), the last term of (6) may be ignored. The dispersion relationship may then be approximated by

$$\omega_i^2 \cong \left(\frac{1}{b_+ + b_-}\right)\omega_p^2\left(\frac{l\tau}{R}\right)\frac{(\beta_0^2+2h)}{(1+\gamma_0^2h^2)}. \tag{8}$$

Thus, the sufficient condition for stability is

$$h < -\beta_0^2/2 \tag{9}$$

for the usual case[13] $b_+ + b_- > 0$. This stability condition (9), together with the definition of h in (5), implies that a sufficiently strong, radially inward electric field may render the relativistic electron beam stable against the negative mass instability. This stabilization is independent of the beam velocity spread or betatron motion, and since its derivation has already taken into consideration the dc self-field effects, the criterion (9) is not restricted to a low-current beam.[16] In the event that the externally imposed electric field exceeds the self-fields, the stability condition (9) may be rewritten as

$$|e\phi| > \tfrac{511}{2}\beta_0^4\gamma_0^3 \ln(b/a), \tag{10}$$

where $|e\phi|$ is the externally imposed potential difference (in kiloelectronvolts) between $r = a$ (the cathode) and $r = b$ (the anode).

As an example, take $R = 100$ cm, $b - a = 4$ cm. Then, according to (10), a 1-MeV electron beam would be stable against the negative mass instability if the inner conductor is negatively biased at a voltage greater than 200 keV with respect to the outer conductor.

A partial explanation of the stability condition (9) may be given in terms of the single particle motion in an *externally* imposed field E_0 and B_0. Let ϵ be the total energy (kinetic and potential) of an electron. One may easily deduce from (1) that $d\omega_0/d\epsilon = (d\omega_0/dr)dr/d\epsilon \propto (\beta_0^2+2h)/(1+\gamma_0^2h^2)$ if the self-fields of the electron layer are neglected. Thus, the effective azimuthal inertia[1,9] of a rotating relativistic electron may be converted from negative to positive if h is less than $-\beta_0^2/2$. It should be stressed, however, that the stability condition (9) is derived from collective-mode considerations which include *both* the ac and the dc self-fields.[17]

In summary, this Letter presents a novel, robust method to suppress a major instability in circular accelerators. Technical aspects such as fabrication, beam injection, and beam retrieval remain to be studied. A more refined analysis may be needed to examine the possible occurrence (if any) of residual instabilities. The stability criterion (9) may be tested on several currently operating devices.

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[13]The expressions for b_+ and b_- for the present geometry are given by Eqs. (49) and (50) of Ref. 8, where it is shown that $1/[l\epsilon_0(b_+ + b_-)]$ is equivalent to

the geometrical g factor for the toroidal configuration treated in Ref. 1. For the toroidal configuration, g is always positive.

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[16]If the container wall is lossy, the wave admittance $(b_+ + b_-)$ would be complex. The electron beam may then be subject to resistive wall instabilities even if the stability criterion (9) is satisfied. The resistive growth rate and the negative mass growth rate scale differently, however. We wish to thank A. M. Sessler (private communication) for reminding us of the importance of resistive wall instabilities, and for furnishing an argument supporting our conclusion on the stabilization mechanism.

[17]It is of some interest to note that the negative mass factor $d\omega_0/d\epsilon$ is maximized with respect to h when $h = 1/\gamma_0^2$. Thus, according to Eqs. (1) and (5), the E layer is most unstable, and is therefore most likely to yield radiation, if its equilibrium rotation is *solely* supported by a radially outward electric field. Reference 11 reported a potent radiation source of this type. Moreover, since the dispersion relationship (8) is applicable for arbitrary combinations of E_0 and B_0, and for arbitrary energy of the electron beam, it provides a ready comparison of the "potency" among various microwave devices such as the gyrotron, orbitron, heliotron, and cross-field devices (if the small-signal growth rate is used as a criterion). Further discussions, as well as the confirmation of the stability criterion (9) by a numerical integration of (4), will be reported elsewhere.