Observation of a Nonpropagating Hydrodynamic Soliton

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When a trough resonator partially filled with water is parametrically driven at an appropriate frequency and amplitude, one or more surface-wave solitons with polaronlike behavior are created. Their properties and interactions are described.

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We report on the experimental discovery of a nonpropagating hydrodynamic soliton, and describe its properties including the interaction between solitons. It is a surface wave in water contained in a trough resonator undergoing continuous excitation.

The apparatus is quite simple. We use a Plexiglas channel 38 cm long and 2.54 cm wide filled with water to a depth of 2 cm. Several drops of the wetting agent Kodak Photo-Flo are added to minimize surface pinning at the walls. Waves are generated by placing the horizontal trough on a loudspeaker whose cone is driven at a frequency 2ν of about 10 Hz in the vertical direction. Below a certain threshold there is little or no significant wave generation. Above the threshold there is a parametrically excited wave[1,2] at half the drive frequency.

Suppose the trough were very long so that it can be regarded as a waveguide with width coordinate y and length coordinate x. The depth h controls the gravity wave phase velocity c given by[3]

$$c^2=(g/k)\tanh(kh). \tag{1}$$

The surface height is given by[4]

$$z=z_0\cos(k_y y)\exp[i(k_x x\pm 2\pi\nu t)] \tag{2}$$

where $k_y l_y=q\pi$, l_y is the width, $q=1,2,3,\ldots$, and

$$k^2=k_x^2+k_y^2=(2\pi\nu)^2/c^2. \tag{3}$$

The case $q=0$ is the plane-wave mode, $q=1$ has one velocity antinode between the side walls, $q=2$ has 2, etc. If $\nu<\nu_{\text{cutoff}}=qc/2l_y$ then k_x is imaginary, the amplitude down the waveguide decays exponentially, and energy will not propagate, but is instead reflected back. For a finite waveguide of length l_x,

$$z=z_0\cos(k_x x)\cos(k_y y)\cos(2\pi\nu t)$$

where $k_x l_x=p\pi$ and $p=0,1,2,3,\ldots$. Resonances are given by

$$\nu_{p,q}=\frac{c}{2}\left[\left(\frac{p}{l_x}\right)^2+\left(\frac{q}{l_y}\right)^2\right]^{1/2}. \tag{4}$$

The $(p,q)=(0,1)$ mode is the equivalent of being at the first cutoff for the infinite waveguide.

We find that we can observe all the low-lying $(p,0)$ modes with the parametric drive. If instead of the vertical parametric excitation we excite the resonator by oscillating it horizontally in the y direction, we find that $\nu_{0,1}$ is a function of the response amplitude, being lower the higher the amplitude. It decreases by 20% when the peak height increases from a very small value to 2 cm. When we parametrically excite the (0,1) mode, however, it appears to be unstable.

What we observe instead of the (0,1) mode are one or more excitations highly localized in the x direction, "sloshing" in the y direction. It is suprising and intriguing that while all parts of the trough are oscillated with equal amplitude, the vigorous wave motion occurs in a space of only a few centimeters long while the rest of the water remains quiet. The phase shift along x given by Eq. (2) for $\nu_{\text{cutoff}}=5.1$ Hz and $\nu=5.4$ Hz is 90° in a distance of 6 cm. Yet, within our resolution of 3°, we find no measurable phase difference between any two points in the wave. We believe that this absence of a phase difference is of fundamental importance; the wave is not simply a waveguide mode occurring slightly above the cutoff frequency but a new type of nonpropagating solitary wave. As we continue we shall see that it has properties usually associated with solitons and shall, accordingly, use this term in describing its behavior. Figure 1 is a computer-generated profile of the soliton based on measurements taken from photographs, when it is at its peak on the far side of the trough. Half a period later the wave peak will have the same height on the near side of the trough. The wave frequency here is 5.1 Hz.

Essentially identical phenomena have been observed in an annular resonator 72 cm in mean circumference and 2.2 cm wide, and in a straight resonator 19 cm long. With a horizontal drive on a linear resonator we have also seen the soliton, accompanied, however, by a low-amplitude (0,1)

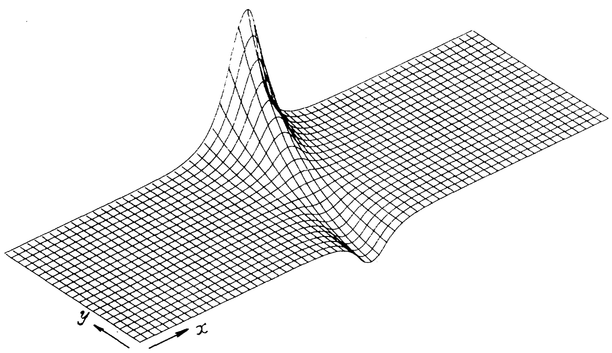

FIG. 1. A plot of

$$z = \mathrm{sech}(x/1.12)[2.8\exp(-1.1y) - 0.70] \times \cos(2\pi \times 5.1t) \text{ cm},$$

at $t = 0$, which is a profile of the soliton based on curve fits to measurements taken from photographs.

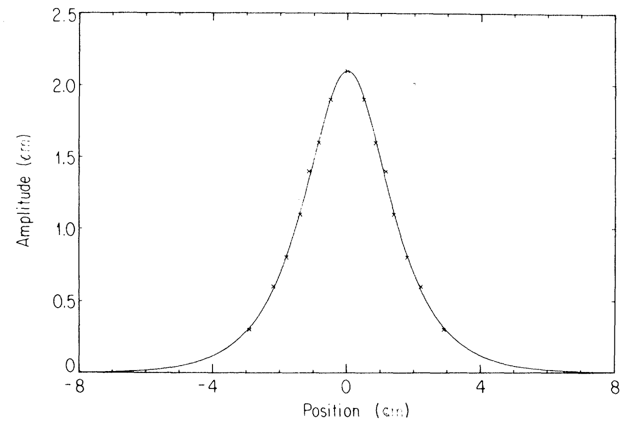

FIG. 2. The height of the soliton as a function of x. The points are experimental data, and the solid curve is $z = 2.1\,\mathrm{sech}(x/1.12)$ cm.

mode. All this suggests that neither the parametric drive nor the resonator ends are central to understanding the phenomena.

We understand the localization of the soliton as the soliton "digging its own hole." Because $\nu_{\mathrm{cutoff}} = \nu_{0,1}$ is depressed at higher amplitudes, $\nu > \nu_{\mathrm{cutoff}}$ in the center of the soliton but $\nu < \nu_{\mathrm{cutoff}}$ in the wings. Energy cannot escape from the center; it is reflected back at its evanescent wings. This self-trapping is reminiscent of polaron behavior. Dr. L. A. Turkevich called the soliton a hydrodynamic polaron. We have also trapped solitons by placing a mound on the bottom of the trough so that there is a localized shallow.

Figure 2 is a plot of the height z of the soliton as a function of x. The points are experimental and the curve is $z = 2.1\,\mathrm{sech}(x/1.12)$ cm, a functional dependence which characterizes some solitions.[5] At the tail of the hyperbolic secant the decay length, 1.12 cm, is equal to the low-amplitude waveguide evanescence length γ, and this is just the magnitude of k_x^{-1}. Thus we can determine the phase velocity of this wave from

$$c = \frac{\omega}{(k_y^2 + k_x^2)^{1/2}} = \frac{2\pi(5.1)}{[(\pi/l_y)^2 - 1/\gamma^2]^{1/2}} = 37.7 \text{ cm/sec}.$$

Using Eq. (1) with a correction for surface tension[3] we find the value 33.7 cm/sec.

The dependence of z on y when the wave is at its peak was found to be far from a cosine function. A best-fit curve is $z = 2.8\exp(-1.1y) - 0.70$ cm.

If we slightly tilt the trough the soliton slowly moves toward the shallow end. For a slope of 0.05, for instance, the soliton's average speed is 0.05 cm/sec. When the trough and loudspeaker are accurately horizontal the soliton maintains its x position in the trough for times the order of an hour or more. We hypothesize that this stability is due to pinning effects at the walls.

There is generally some degree of competition between the various modes of the trough. We have devised ways to selectively encourage the appearance of the solitons to the exclusion of competitive modes. One obvious way is to choose trough dimensions which separate in frequency the unwanted modes from the soliton mode. Another very successful procedure is to nucleate the soliton mode by producing a disturbance which is compatible with it. Thus sloshing motion across the width can be produced by rocking the resonator or with the help of a hand-held paddle. This can be very effective if it is done when the drive is first turned on.

Since the frequency of the wave is half that of the drive, the phase of the drive is repeated every 180° of the wave. Consequently solitons which are 0° or 180° apart in phase are compatible and equally driven. Solitons which are 180° out of phase can be generated by sharply twisting the trough about a central vertical axis, so that the sloshing in the left half has the opposite direction of that in the right half. Alternatively an inverted T-shaped paddle immersed in the water can be twisted to achieve the same result.

Solitons can be moved in various ways. We have already mentioned tilting the trough. They move in response to gentle jets of air. They can also be nudged by rods of soft sponge plastic. Unwanted solitons are killed by stabbing them with such rods.

The dashed lines of Fig. 3 describe the range of drive amplitude and drive frequency in which individual solitons are observed without hysteresis. The eight full-line curves are equal–wave-response curves. The number on each is the peak height in centimeters of the soliton above the equilibrium level of the water. Note that at 0.7 and 2.1 cm the curves are so short that they are only represented by points. Outside the dashed boundary, changes in drive amplitude and/or frequency are generally hysteretic. If it exists, the soliton may be unstable, there may be appreciable x-dependent phase shifts, and there may be a mixture of modes. At a drive amplitude exceeding 0.081 cm (peak to peak), characteristically a competing mode rather than a soliton appears.

As pointed out earlier, we can generate more than a single soliton in the resonator with the same or opposite polarity. We want now to describe the interaction of two such solitons. Two solitons of the same polarity attract each other, but only weakly if the distance between them is significantly larger (say a factor of 3) than their effective length. Two solitons which start out 20 cm apart center to center, for example, take about 15 min to reach a separation where they strongly overlap. When this happens their attractive speed greatly increases.

If the frequency is substantially lower than the small amplitude $\nu_{0,1}$, the two solitons combine; the end result is a single soliton having the amplitude of each of the initial solitons. If the frequency is closer to the small amplitude $\nu_{0,1}$, the solitons oscillate about each other. Figure 4 is a schematic of the stages of oscillation. In the first stage, 4(a), there is significant overlap. In Fig. 4(b) the overlap is sufficient to create an amplitude at the center which is comparable with the amplitudes of solitons 1 and 2. In Fig. 4(c) the solitons completely overlap. In Fig. 4(d) apparently solitons 1 and 2 have passed through each other and exchanged places while in Fig. 4(e) the configuration is the original one. Left undisturbed this sequence repeats indefinitely. We have observed it for periods of more than an hour. We have never seen the oscillation interrupted except by accident or design.

A simple transducer which responds to wave heights is a pair of vertical wire electrodes that dip into the water. At a constant-voltage drive (ac or dc) the current due to the conductance of the water is proportional to the wave height at the electrodes. Figure 5 shows the response of such a transducer placed at the midpoint of the oscillating solitons of Fig. 4. In the lowest trace, which covers the complete sequence of Fig. 4, the paper speed was great enough so that the individual oscillations are seen ($\nu = 5.32$ Hz). The frequency of the envelope, which is much deeper than the drive, is $2\nu/$

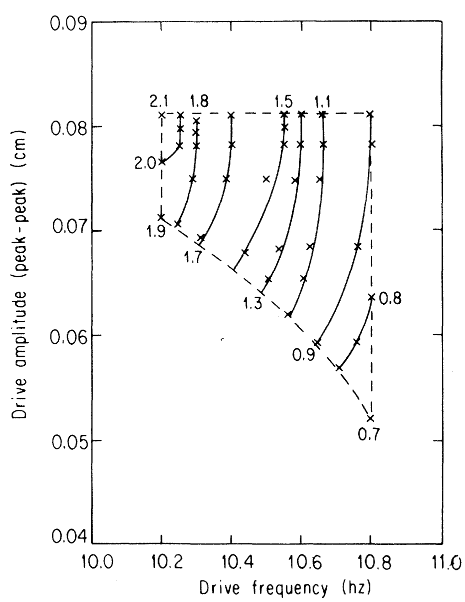

FIG. 3. The range of drive amplitude and drive frequency in which individual solitons are observed without hysteresis. The dashed lines are the boundaries of this region. The eight full-line curves are equal–wave-response curves. The number associated with each curve is the peak height in centimeters of the solitons above the equilibrium level of the water.

FIG. 4. A schematic of stages in the resulting oscillation of two like-phase solitons. (a) Significant overlap between soliton 1 and soliton 2. (b) Sufficient overlap to create an amplitude at the center comparable to the amplitude of solitons 1 and 2. (c) The solitons completely overlap. (d) Solitons 1 and 2 have passed through each other and exchanged places. (e) The original configuration. Solitons 1 and 2 have exchanged places.

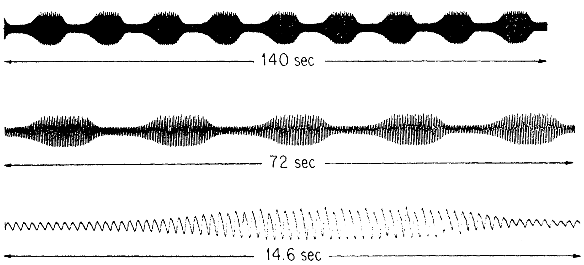

FIG. 5. The response of a height transducer placed at the midpoint of the oscillating solitons of Fig. 4. The lowest trace covers the complete sequence of Fig. 4. The frequency of the envelope is $2\nu/(156 \pm 0.5) = 0.068$ Hz. The upper traces are recorded at successively slower paper speeds.

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A pair of solitons of opposite polarity in close proximity to each other repel each other and slowly move until they are approximately 12 cm apart, and then maintain this separation indefinitely. Again we have never seen this state to be interrupted except by accident or design. We understand the attraction and repulsion of solitons in terms of the Bernoulli effect. In the overlap region the particle velocity increases for the like pair and decreases for the unlike pair, decreasing or increasing, respectively, the pressure between them.

By introducing tracers (eccospheres) into the water we have been able to establish that the motion of solitons is one in which there is a transport of excitation with negligible transport of mass.

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