Kac-Moody Symmetries of Kaluza-Klein Theories

L. Dolan
The Rockefeller University, New York, New York 10021

and

M. J. Duff
The Blackett Laboratory, Imperial College, London SW7 2BZ, England

(Received 12 September 1983)

The authors identify symmetries of the four-dimensional Lagrangian, including the massive states, obtained via Kaluza-Klein compactification from gravity in five dimensions and ground state $M^4 \times S^1$. The symmetries are described by (1) a Kac-Moody extension of the Poincaré algebra, (2) a Virasoro algebra of internal symmetries including the Salam-Strathdee SO(1,2), and (3) a mixing of (1) and (2). All symmetries are spontaneously broken save for Poincaré ⊗ U(1). The Goldstone bosons provide masses for the spin-2 tower. Higher dimensions and supergravity are discussed.

PACS numbers: 11.30.Ly, 11.30.Qc

Four-dimensional theories obtained from Kaluza-Klein[1] compactification of higher-dimensional gravity theories describe a finite number of massless states and an infinite tower of massive states. The four-dimensional symmetries will depend on the general covariance and other symmetries of the higher-dimensional theory, and the corresponding transformations on the fields will mix the massless and massive sectors. In this Letter, we show that the symmetries are described by an infinite-parameter Kac-Moody[2,3]-type algebra.[4]

For simplicity, we begin by considering pure gravity in five dimensions, with coordinates $Z^M = (x^\mu, \theta/m)$, where $M = 1, \ldots, 5$ and $\mu = 1, \ldots, 4$, described by the action

$$S = (2\pi\kappa^2)^{-1} \int d^4x\, d\theta\, (-\gamma)^{1/2} R(\gamma), \tag{1}$$

which is invariant under the infinitesimal general coordinate transformations

$$Z^M \to Z'^M = Z^M - \zeta^M(Z). \tag{2}$$

We are assuming that the ground state is given by $M^4 \times S^1$, i.e., four-dimensional Minkowski space times a circle of radius m^{-1}, and hence that $0 \leqslant \theta \leqslant 2\pi$. Consequently, in the change of variables given by

$$\gamma_{MN}(Z) = \varphi^{-1/3} \begin{bmatrix} g_{\mu\nu} + \kappa^2 \varphi A_\mu A_\nu & \kappa\varphi A_\mu \\ \kappa\varphi A_\nu & \varphi \end{bmatrix}, \tag{3}$$

the fields $g_{\mu\nu}$, A_μ, and φ are periodic in θ and may be Fourier expanded in the form

$$\begin{aligned} g_{\mu\nu}(x,\theta) &= \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x) e^{in\theta}, \\ A_\mu(x,\theta) &= \sum_{n=-\infty}^{\infty} A_{\mu n}(x) e^{in\theta}, \\ \varphi(x,\theta) &= \sum_{n=-\infty}^{\infty} \varphi_n(x) e^{in\theta} \end{aligned} \tag{4}$$

with $g_{\mu\nu n}{}^* = g_{\mu\nu\, -n}$, etc.

It is well known[5] that, after integrating over θ and retaining only the $n=0$ terms in Eq. (4), one obtains a theory of a massless spin 2, $g_{\mu\nu 0}$; a massless spin 1, $A_{\mu 0}$; and a massless spin 0, φ_0, described by the action

$$S = \int d^4x (-g_0)^{1/2} \left(\frac{R(g_0)}{\kappa^2} - \frac{1}{4} \varphi_0 F_{\mu\nu 0} F^{\mu\nu}{}_0 - \frac{1}{6\kappa^2} \frac{\partial_\mu \varphi_0 \partial^\mu \varphi_0}{\varphi_0{}^2} \right), \tag{5}$$

where indices are raised and lowered by $g_{\mu\nu 0}$ and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The action is invariant under general coordinate transformations

$$\begin{aligned} \delta g_{\mu\nu 0} &= \partial_\mu \zeta_0{}^\rho g_{\rho\nu 0} + \partial_\nu \zeta_0{}^\rho g_{\rho\mu 0} + \zeta_0{}^\rho \partial_\rho g_{\mu\nu 0}, \\ \delta A_{\mu 0} &= \partial_\mu \zeta_0{}^\rho A_{\rho 0} + \zeta_0{}^\rho \partial_\rho A_{\mu 0}, \\ \delta\varphi_0 &= \zeta_0{}^\rho \partial_\rho \varphi_0; \end{aligned} \tag{6}$$

local gauge transformations,

$$\delta A_{\mu 0} = \kappa^{-1} \partial_\mu \zeta_0{}^5; \tag{7}$$

and global scale transformations,

$$\delta A_{\mu 0} = \lambda A_{\mu 0}; \quad \delta\varphi_0 = -2\lambda\varphi_0. \tag{8}$$

The symmetry of the vacuum, determined by the vacuum expectation values

$$\langle g_{\mu\nu 0}\rangle = \eta_{\mu\nu}, \quad \langle A_{\mu\nu}\rangle = 0, \quad \langle \varphi_0 \rangle = 1, \tag{9}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, is the four-dimensional Poincaré group ⊗ R. Thus the masslessness of $g_{\mu\nu 0}$ is due to general covariance and the masslessness of $A_{\mu 0}$ to gauge invariance, but φ_0 is massless because it is the Goldstone boson associated with the spontaneous breakdown of the global scale invariance. Note that the gauge group is R rather than U(1) because this truncated $n=0$ theory has lost all memory of the periodicity in θ.

In this paper, however, we wish to analyze the four-dimensional symmetries of the theory which results from retaining the $n\neq 0$ modes in Eq. (4) and which describes in addition to the above massless states an infinite tower of charged, massive, purely spin-2 particles[6] with charges $q_n = nKm$ and masses $m_n = |n|m$. To this end, we also Fourier expand the general coordinate parameter ζ^M of Eq. (2) in the form

$$\zeta^\mu(x,\theta) = \sum_{n=-\infty}^{\infty} \zeta_n{}^\mu(x)e^{in\theta}, \tag{10}$$

$$\zeta^5(x,\theta) = \sum_{n=-\infty}^{\infty} \zeta_n{}^5(x)e^{in\theta} \tag{11}$$

with $\zeta_n{}^{M*} = \zeta_{-n}{}^M$. An important observation is that the assumed topology of the ground state, namely $M^4\times S^1$, restricts us to general coordinate transformations periodic in θ. Whereas the general covariance of Eq. (6) and the local gauge invariance of Eq. (7) simply correspond to the $n=0$ modes in Eqs. (10) and (11), respectively, the global scale transformation of Eq. (8) is no longer a symmetry because it corresponds to a rescaling $\Delta\gamma_{MN0} = (2\lambda/3)\gamma_{MN0}$ together with a general coordinate transformation of the form

$$\zeta^5 = -\lambda\theta/m, \tag{12}$$

which is now forbidden by the periodicity requirement. The field φ_0 is therefore merely a pseudo-Goldstone boson and its masslessness an artifact of the tree approximation.

Just as ordinary general coordinate invariance may be regarded as the local gauge symmetry corresponding to the global Poincaré algebra obtained from the restriction

$$\zeta^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu, \tag{13}$$

where a^μ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are constants, so the infinite-parameter local transformation of Eqs. (10) and (11) corresponds to an infinite-parameter global algebra. To determine its properties we make an analogous restriction

$$\zeta_n{}^\mu(x) = a_n{}^\mu + \omega^\mu{}_{\nu n} x^\nu, \tag{14}$$

$$\zeta_n{}^5(x) = c_n, \tag{15}$$

where $a_n{}^\mu$, $\omega^\mu{}_{\nu n}$, and c_n are constants. The corresponding generators are

$$P_n{}^\mu = e^{in\theta}\partial^\mu, \tag{16}$$

$$M_n{}^{\mu\nu} = e^{in\theta}(x^\nu\partial^\mu - x^\mu\partial^\nu), \tag{17}$$

$$Q_n = ie^{in\theta}\partial_\theta, \tag{18}$$

and they define the following noncompact infinite-parameter Lie algebra:

$$[P_n{}^\mu, P_m{}^\nu] = 0, \tag{19}$$

$$[M_n{}^{\mu\nu}, P_m{}^\sigma] = \eta^{\mu\sigma}P_{n+m}{}^\nu - \eta^{\nu\sigma}P_{n+m}{}^\mu, \tag{20}$$

$$[M_n{}^{\mu\nu}, M_m{}^{\rho\sigma}] = \eta^{\mu\rho}M_{n+m}{}^{\nu\sigma} + \eta^{\nu\sigma}M_{n+m}{}^{\mu\rho} - \eta^{\nu\rho}M_{n+m}{}^{\mu\sigma} - \eta^{\mu\sigma}M_{n+m}{}^{\nu\rho}, \tag{21}$$

$$[Q_n, Q_m] = (n-m)Q_{n+m}, \tag{22}$$

$$[Q_n, P_m{}^\mu] = -mP_{n+m}{}^\mu, \tag{23}$$

$$[Q_n, M_m{}^{\mu\nu}] = -mM_{n+m}{}^{\mu\nu}. \tag{24}$$

Since $n, m = -\infty, \dots, -1, 0, 1, \dots, \infty$, Eqs. (19), (20), and (21) define a Kac-Moody–like loop algebra, Poincaré $\otimes\, C[t, t^{-1}]$. Equation (22) is the Virasoro algebra without a central extension. Equations (23) and (24) describe a mixing between the generalized "internal" generators Q_n and the generalized "space-time" generators $P_n{}^\mu$ and $M_n{}^{\mu\nu}$. It will be interesting to see whether the absence of the central extension in this algebra persists in the quantum theory.

Although the above algebra is a symmetry of the four-dimensional Lagrangian, the symmetry of the vacuum determined by the vacuum expectation values

$$\langle g_{\mu\nu}\rangle = \eta_{\mu\nu}, \quad \langle A_\mu\rangle = 0, \quad \langle\varphi\rangle = 1 \tag{25}$$

is only Poincaré $\otimes$ U(1). Consequently, neither the Kac-Moody extension of the Poincaré group nor the mixing of internal and space-time symmetries of Eqs. (23) and (24) violates the Coleman-Mandula[7] theorem which states that the symmetries of the S matrix should be of the direct-product form Poincaré $\otimes\, G$. The finite-dimensional subalgebra of Eqs. (19) to (24) may, however, be enlarged to Poincaré $\otimes$ SO(1, 2) since $P_0{}^{\mu}$, $M_0{}^{\mu\nu}$, Q_{-1}, Q_0, Q_1 also close. This SO(1, 2) symmetry was already observed by Salam and Strathdee.[6]

Since the gauge parameters $\zeta_n{}^{\mu}(x)$ and $\zeta_n{}^5(x)$ each correspond to spontaneously broken generators except for $n=0$, it follows that for $n\neq 0$, the fields $A_{\mu n}(x)$ and $\varphi_n(x)$ are the corresponding Goldstone boson fields. The corresponding gauge fields $g_{\mu\nu n}$, with two degrees of freedom, will then each acquire masses by absorbing the two degrees of freedom of each vector Goldstone boson on $A_{\mu n}$ and the one degree of freedom of each scalar Goldstone boson φ_n to yield a pure spin-2 massive particle with five degrees of freedom. This accords[6,8] with the observation that the massive spectrum is purely spin 2. In a somewhat different context, the spontaneous breakdown of general covariance and the appearance of massive spin-2 particles via vector and scalar Goldstone bosons has been observed before.[9]

To see that the transformations on the fields induced by Eqs. (10) and (11) mix the massless and massive modes, we calculate the kth transformation on the nth fields $g_{\mu\nu n}$, $A_{\mu n}$, φ_n:

$$\Delta_k\varphi_n=\zeta_k{}^{\rho}\partial_{\rho}\varphi_{n-k}+im(n+2k)\zeta_k{}^5\varphi_{n-k}+\kappa(3imk)\zeta_k{}^{\rho}\sum_{l=-\infty}^{\infty}\varphi_{n-k-l}A_{\rho l},$$

$$\Delta_k A_{\mu n}=\kappa^{-1}\Big(\delta_{nk}\,\partial_{\mu}\zeta_k{}^5+imk\zeta_k{}^{\rho}\sum_{l=-\infty}^{\infty}\varphi_{n-l-k}{}^{-1}g_{\mu\rho l}\Big)+\partial_{\mu}\zeta_k{}^{\rho}A_{\rho(n-k)}+\zeta_k{}^{\rho}\partial_{\rho}A_{\mu(n-k)}$$
$$+im(n-2k)\zeta_k{}^5A_{\mu(n-k)}-\kappa(imk)\zeta_k{}^{\rho}\sum_{l=-\infty}^{\infty}A_{\mu(n-k-l)}A_{\rho l},\qquad(26)$$

$$\Delta_k g_{\mu\nu n}=\partial_{\mu}\zeta_k{}^{\rho}g_{\rho\nu(n-k)}+\partial_{\nu}\zeta_k{}^{\rho}g_{\rho\mu(n-k)}+\zeta_k{}^{\rho}\partial_{\rho}g_{\mu\nu(n-k)}+imn\zeta_k{}^5g_{\mu\nu(n-k)}$$
$$+\kappa(imk)\zeta_k{}^{\rho}\sum_{l=-\infty}^{\infty}\{g_{\mu\nu(n-k-l)}A_{\rho l}-g_{\mu\rho(n-k-l)}A_{\nu l}-g_{\nu\rho(n-k-l)}A_{\mu l}\}.$$

Here $\varphi_n{}^{-1}$ is defined by $\varphi^{-1}=\sum_{n=-\infty}^{\infty}\varphi_n{}^{-1}e^{in\theta}$. In Eq. (26), the zero-mode transformations ($k=0$) do not mix fields of different mass or spin. The higher-k transformations, however, do mix the massless ($n=0$) fields with the higher-n fields. Note also, as expected from the mixing of the internal and space-time generators, that they mix fields of different spin.

Although, for simplicity, we have confined our attention to the case of pure gravity in five dimensions, infinite-parameter symmetries of the kind discussed here will also appear when we consider more complicated theories in higher dimensions and when the symmetry of the extra dimensions is non-Abelian. There is a considerable increase in complexity, however, as may be seen from the simplest example of $S^2=\mathrm{SO}(3)/\mathrm{SO}(2)$. The scalar harmonics $e^{in\theta}$ on S^1 will now be replaced by the spherical harmonics $Y_m{}^l(\theta,\varphi)$ and the structure constants of the infinite-parameter algebra will be determined by the Clebsch-Gordan coefficients. An understanding of the Kac-Moody symmetries in these models will no doubt be important in analyzing the question of ultraviolet divergences.

Finally, super–Kac-Moody algebras will make their appearance when one considers the Fourier expansion of local supersymmetry parameters[4] in more dimensions. Of particular interest will be those which relate the massive $N=8$ supermultiplets[10] which arise from Kaluza-Klein compactification of $d=11$ supergravity on the seven-sphere. The fact that the scalar fields which appear massless at tree level in Kaluza-Klein theories are at best only pseudo-Goldstone bosons presumably means that their masslessness is an artifact of the tree approximation[11] unless they are the superpartners of gauge fields in an extended $N>1$ supersymmetry multiplet, or perhaps of chiral spin $\frac{1}{2}$ in $N=1$.

One of us (M.J.D.) is grateful for the hospitality extended to him by the theory group at Rockefeller University. This work was supported in part by the U. S. Department of Energy under Contract No. DE-AC02-81ER40033B.000.

[1]Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin Math. Phys. Kl. 966 (1921); O. Klein, Z. Phys. 37, 895 (1926).

[2]V. G. Kac, Izv. Akad. Nauk SSSR Ser. Mat. 32, 1323 (1968) [Math. USSR Izv. 2, 1271 (1968)]; R. Moody, J. Algebra 10, 211 (1968).

[3]Kac-Moody algebras as symmetries of particle

theories were first identified by L. Dolan, Phys. Rev. Lett. 47, 1371 (1981), and Phys. Lett. 113B, 387 (1982), and Rockefeller University Report No. RU83/B/63 (to be published).

[4]The existence of an infinite-parameter symmetry in Kaluza-Klein theories was pointed out by M. J. Duff, Nucl. Phys. B219, 389 (1983), and in Proceedings of the Marcel Grossman Meeting, Shanghai, August 1982 (to be published). See also B. Biran, F. Englert, B. deWit, and H. Nicolai, Phys. Lett. B 124, 45 (1983). A different use of Kaluza-Klein theories to discuss Kac-Moody symmetry in the Yang-Mills sector is given by L. Dolan, Rockefeller University Report No. RU83/B/55, in Proceedings of Orbis Scientiae, Coral Gables, 1983 (to be published).

[5]T. Appelquist and A. Chodos, Phys. Rev. Lett. 50, 141 (1983), and Phys. Rev. D 28, 772 (1983).

[6]A. Salam and J. Strathdee, Ann. Phys. (N.Y.) 141, 316 (1982).

[7]S. Coleman and J. Mandual, Phys. Rev. 159, 1251 (1967).

[8]This is well known in the literature. See, for example, A. Karlhede and T. Tomaras, Phys. Lett. 125B, 49 (1983).

[9]M. J. Duff, Phys. Rev. D 12, 3969 (1975).

[10]M. J. Duff, B. E. W. Nilsson, and C. N. Pope, Phys. Rev. Lett. 50, 2043 (1983).

[11]One of us (M.J.D.) is grateful for conversations with E. Cremmer, H. Osborn, S. Weinberg, and E. Witten on this point.