Effective-Spin Model for Finite-Temperature QCD

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An effective-spin model for finite-temperature QCD is derived by use of a variant of the Migdal-Kadanoff renormalization group. Mean field theory and [for SU(3)] Monte Carlo simulation are applied to the model in order to find the deconfining phase transition. The results are in very good agreement with Monte Carlo simulations of the full theory. Mean field theory predicts that the deconfining transition in pure SU(N) lattice gauge theory is first order for all $N \geqslant 3$.

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There has been a great amount of interest recently in the deconfining phase transition at finite temperature in QCD. An important advance was the work of Svetitsky and Yaffe,[1] which pointed out that it was in principle possible to construct from a $(d+1)$-dimensional gauge theory an effective theory in which Wilson lines acted as spins in a d-dimensional spin system. This spin system would have global Z_N invariance, inherited from the underlying gauge theory. They applied renormalization-group ideas from the modern theory of critical phenomena to deduce various properties of the deconfining transition. A second advance is due to Banks and Ukawa,[2] who showed that the effect of fermions on the Wilson lines was like that of an external magnetic field on a spin system.

In this Letter I will derive from lattice gauge theory an effective-spin model which describes the finite-temperature behavior of Wilson lines. This will be done with use of a variant of an approximate real-space renormalization group due to Midgal and Kadanoff.[3] The effect of fermions will be included. A brief discussion of the results which can be derived from this effective-spin model is included.

The Wilson form of an SU(N) lattice gauge theory in $(d+1)$ dimensions is given by

$$A=\sum\nolimits_p (\beta/2N)\mathrm{Tr}[U(\partial p)+\mathrm{H.c.}], \tag{1}$$

where β is equal to $2N/g^2$. Finite temperature is introduced into the Feynman path integral by imposing periodic boundary conditions in the timelike direction, with period given by $1/T$, where T is temperature. Thus, $T=1/na$, where a is the lattice spacing and n is the timelike extent of the lattice in lattice units.

The maximum temperature in unphysical lattice units is given by setting $n=1$. This case provides an explicit, exact example of Wilson lines as SU(N)-valued spins in d dimensions. Let us denote $U_0(i,i)$, the timelike link variable connecting i to i, as $S(i)$. The action can be written as

$$A=\sum_{\text{s.p.}}(\beta/2N)\mathrm{Tr}[U(\partial p)+\mathrm{H.c.}]+\sum_{\langle ij\rangle}(\beta/2N)\mathrm{Tr}[S^\dagger(j)\,U(j,i)S(i)\,U(i,j)+\mathrm{H.c.}]. \tag{2}$$

This is precisely the action of an adjoint representation Higgs model in d dimensions. The dynamics are particularly simple for Abelian models: The spin and gauge degrees of freedom decouple.

If the case $n\neq 1$ could be reduced to the case $n=1$, this would be a powerful simplification. This can be done, albeit approximately by use of a variant of the Migdal-Kadanoff real-space renormalization group.[3] The first step is a bond-moving procedure which removes the spatial plaquette (s.p.) interactions from intermediate time slices and shifts the interaction to one time slice. This bond moving always increases the free energy, as the result of an argument based on Jensen's inequality. Kadanoff has given arguments which show that bond moving is a good approximation in the limits of strong and weak coupling.

After bond moving, the integration over intermediate spatial links can be done exactly. In these two limits the action obtained from decimation is approximately of the Wilson form. The result is

$$A=\sum_{\text{s.p.}}(n\beta/2N)\mathrm{Tr}[U(\partial p)+\mathrm{H.c.}]+\sum_{l}(\beta'/2N)\mathrm{Tr}[U^\dagger S^\dagger US+\mathrm{H.c.}], \tag{3}$$

where $S(i)$ is the Wilson line based at i. The parameter β' is given by

$$C_F(\beta')=[C_F(\beta)/C_0(\beta)]^n, \tag{4}$$

where C_0 and C_F are the coefficients of the identity and fundamental representation characters in the character expansion of the action. The leading behavior of β' in weak and strong coupling is given by

$$\beta' \to \begin{cases} \beta/n, & \beta \text{ large}, \\ \beta^n/(2N^2)^{n-1}, & \beta \text{ small}. \end{cases} \tag{5}$$

The first result has the correct weak-coupling behavior, as can be seen as follows. One can rescale the continuum action so that Euclidean time runs from 0 to 1 instead of 0 to $1/T$. The action becomes

$$S = \frac{1}{g^2 T}\int_0^1 d\tau \int d^3x \, \tfrac{1}{2}\,\mathrm{Tr}(T^2E^2 + B^2). \tag{6}$$

With the identification $T = 1/na$, the correctness of Eq. (3) follows. In the strong-coupling limit, the result is easily verified by a strong-coupling expansion.

It is possible in the two limits of strong and weak coupling to integrate out approximately the remaining spatial links. In the limit $n\beta << 1$, this can be done by a straightforward strong-coupling expansion. In the weak-coupling limit $\beta \to \infty$, one can integrate over all gauge copies of the identity, i.e., over all $U(i,j)$ of the form $g(i)g^\dagger(j)$. Amusingly, the answer is the same in both cases. The final result is an effective action which involves only the Wilson lines:

$$A = \sum_{\langle ij \rangle} J[\mathrm{Tr}S(i)\mathrm{Tr}S^\dagger(j) + \mathrm{H.c.}], \tag{7}$$

where

$$J = \frac{\beta'}{2N^2} \to \begin{cases} \beta/2N^2 n, & \beta \text{ large}, \\ (\beta/2N^2)^n, & \beta \text{ small}. \end{cases} \tag{8}$$

This action could have been conjectured on general grounds: It is the simplest local action with the required Z_N invariance. Note that although the "spins" $S(i)$ are SU(N) valued, the action does not have the conventional SU(N) $\otimes$ SU(N) invariance. This is necessary, for the spins transform as the adjoint representation under local gauge transformations. Thus the action can only depend on $\mathrm{Tr}S(i)$.

The transformation performed above is most appropriate if n is small, typically 2 or 3. It does not exhibit the correct scaling relationship between β and n in the weak-coupling region. In most finite-temperature Monte Carlo simulations performed to date, these values of n have been used. For much larger values of n and β, it is more appropriate to use the full Migdal-Kadanoff renormalization group. This will give a Higgs model for which the correct scaling behavior will be manifest, but the quantitative relationship between β and n will be wrong. This reflects a well-known property of the Migdal-Kadanoff renormalization group.

Fermions can be included in this procedure in a straightforward way. The action for the Wilson fermions is given by

$$A_F = \sum_{n,\mu} K\{\bar\psi(n)(1-\gamma_\mu)U_\mu^\dagger(n)\psi(n+\mu)\bar\psi(n+\mu)(1+\gamma_\mu)U_\mu(n)\psi(n)\} + \sum_n \bar\psi(n)\psi(n), \tag{9}$$

where K is the fermion hopping parameter. It will be useful later to distinguish the hopping parameter for the timelike direction, K_t, from K_s, the hopping parameters for the spatial directions. It is easy to include a fermion chemical potential, but this will not be done here. The first step in handling the fermions is a moving of the spatial hopping terms on intermediate time slices to the first time slice. The inclusion of fermions destroys the variational character of the bond moving, but this will not be important. There is an ambiguity in the treatment of the fermion mass term. Some portion of the mass term in the action can be moved with the hopping terms; the amount is arbitrary. In this case I will move none of the mass term. This helps insure that correct results are obtained in the large mass limit of small K. After the spatial hopping terms are moved, the integration over the fermion fields on intermediate time slices can be performed exactly. The result is a fermionic action of the same form, with K_s' and K_t' given by

$$K_s' = nK, \quad K_t' = 2^{n-1}K^n. \tag{10}$$

The timelike and spatial hopping terms have rather different effects on the effective-spin system. For the moment, I will consider only the timelike terms, which makes the problem trivial. The effect of timelike fermion trajectories is that of an external field applied to the Wilson lines. As first pointed out by Banks and Ukawa,[2] this term explicitly breaks the Z_N symmetry of the effective-spin theory. This term will be the most important effect of fermions if K and n are small.

The effect of the spatial hopping terms is harder to analyze. Roughly speaking, the dominant effect is to induce a long-ranged force between Wilson lines. In the confining phase, this interaction should fall off exponentially with the mass of the lightest scalar meson.

The final result for the effective-spin system action after the inclusion of purely timelike fermion loops has the approximate form

$$A = \sum_{\langle ij \rangle} J[\mathrm{Tr}S(i)\mathrm{Tr}S^{\dagger}(j) + \mathrm{H.c.}] + \sum_i H\,\mathrm{Tr}[S(i) + S^{\dagger}(i)], \tag{11}$$

where H is given by

$$H = 2N_f(2K_t)^n. \tag{12}$$

This form can be generalized in an obvious way to include other terms which presumably occur in the true effective action, such as next-nearest-neighbor interactions.

Mean field theory can be applied to the effective-spin model in a straightforward way. The free energy per site is given by maximizing

$$V = W_0(x) - (8dJ)^{-1}(x - 2H)^2 \tag{13}$$

as a function of x. The function W_0 is the generating function for a single SU(N)-valued spin, given by

$$W_0(x) = \ln\{\int (dU)\exp[\tfrac{1}{2}x\,\mathrm{Tr}(U + U^{\dagger})]\}. \tag{14}$$

Useful formulas for W_0 are given in Ref. 4.

In the case of SU(3), I have also studied this effective-spin model using Monte Carlo simulation. The two methods are in very good agreement on the phase structure of the model, and in reasonable agreement with Monte Carlo simulations of the underlying gauge theory. For SU(2) and SU(3) theories without fermions, the critical values of β are shown in Table I, where they are compared with results of various Monte Carlo simulations of the full finite-temperature gauge theory. In these cases, a strong-coupling form of Eq. (4) was used to determine β'.

It is known from Monte Carlo simulations that the deconfining transition is second order for SU(2) and first order for SU(3). It has been tempting to assume that the transition will be second order for SU(N), if N is sufficiently large. This notion is based on the idea that it is sufficient to consider a effective theory based only on the Z_N part of the Wilson lines. My results indicate that it is more natural to retain the full SU(N) degrees of freedom in the Wilson lines. When mean field theory is applied to the model derived here, it indicates that the deconfining transition is first order for all N greater than or equal to 3. There are theoretical arguments due to Gocksch and Neri[9] based on the lattice Schwinger-Dyson equations which indicate that the large-N deconfining transition is first order. A first-order transition is also consistent with Monte Carlo results from a reduced, large-N model.[10] It seems likely than an effective-spin theory based only on Z_N degrees of freedom does not provide a good representation of the dynamics. This is easy to understand from the effective-spin model. Although $s_i = Z \in Z_N$ maximizes the action, the contribution of such points is strongly suppressed by the group measure.

There is currently some controversy over the effect of fermions on the deconfining transition. It is widely held that sufficiently light quarks will eliminate the phase transition. However, it is an open question how light quarks must be for this to happen. For SU(3), both mean field theory and Monte Carlo simulation indicate that the transition does not occur if H is greater than 0.52. This is in good agreement with the results of Hasenfratz, Karsch, and Stamatescu,[7] who studied the full SU(3) lattice gauge theory for $n = 2$, using Monte Carlo methods for the gauge fields combined with a hopping-parameter expansion for the fermions. They also give a formula for the critical value of β as a function of H. Their result is

$$\beta_c(H) = 5.11 - (4.94 \pm 0.75)H, \tag{15}$$

which should be compared with

$$\beta_c(H) = 4.77 - 4.0H, \tag{16}$$

TABLE I. Critical values of β for SU(2) and SU(3).

n	β theory	β Monte Carlo
	SU(2)	
1	0.676	0.75[a]
2	1.79	1.8[a]
3	2.65	2.15[a]
		1.9[b]
	SU(3)	
1	2.11	
2	4.77	5.11[c]
3	6.14	5.55[d]

[a]Ref. 5. [b]Ref. 6. [c]Ref. 7. [d]Ref. 8.

which I have derived from mean field theory and checked with Monte Carlo simulation.

The results presented here are in very good agreement with the Monte Carlo data, and constitute a strong validation of current theoretical ideas of the deconfining transition. However, these results were derived with use of what is essentially a strong-coupling result. This presumably indicates that current Monte Carlo simulations have been carried out in an intermediate-coupling region. It is therefore very desirable that more Monte Carlo simulations be done at larger values of n, in order to be closer to the continuum limit.

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