Uniqueness Theorem in Scattering Theory

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A reflecting convex obstacle is uniquely defined by the scattering amplitude $f(k_0, \nu_0, n)$ known at a fixed frequency k_0, for a fixed direction ν_0 of the incident wave, and for all directions n of the scattered waves in a solid angle. Earlier the uniqueness theorem was proved under the assumption that $f(k, \nu_0, n)$ is known for $a \leqslant k \leqslant b$, $b > a$, and all n.

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Let $D \subset R^3$ be a strictly convex reflecting bounded obstacle with a smooth boundary Γ and let Ω be the exterior domain. The scattering of a plane wave $u_0 = \exp[ik_0(\nu_0, x)]$ by this obstacle is described by the following problem:

$$(\nabla^2 + k_0^2)u = 0 \text{ in } \Omega, \quad k_0 > 0, \tag{1}$$

$$u = 0 \text{ on } \Gamma, \quad u = u_0 + v, \tag{2}$$

$$v \sim r^{-1}\exp(ikr)f(k_0, \nu_0, n) \text{ as } r \to \infty, \quad n = x/r, \quad r = |x|. \tag{3}$$

The function f is called the scattering amplitude. It is well known that the knowledge of $f(k, \nu_0, n)$ for $a \leqslant k \leqslant b$, $b > a$, and all $n \in S^2$ (where S^2 is the unit sphere in R^3) determines the obstacle D uniquely. The basic result of this note shows that the obstacle is uniquely determined by the values $f(k_0, n) = f(k_0, \nu_0, n)$, $n \in S^2$. This result is of interest not only by itself but also because the inverse source problem is known to have more than one solution which generates the same scattering amplitude $f(k_0, n)$ for a fixed k_0 and all $n \in S^2$.

Theorem.—If $f(k_0, n)$ is known for all $n \in \tilde{S}$, where $\tilde{S} \subset S^2$ is a solid angle, then Γ is uniquely defined.

Proof.—(1) The knowledge of $f(k_0, n)$ for $n \in \tilde{S}$, defines $f(k_0, n)$ uniquely for $n \in S^2$. This is well known, but we give a short proof for convenience of the reader. If $f(k_0, n)$ is known for $n \in \tilde{S}$ then the solution of Eq. (1) is known on a part of the infinitely large sphere and its normal derivative $\partial v/\partial r = ikf(k_0, n)\exp(ik_0 r)/r$ is also known on the same part of this sphere. By the uniqueness of the solution to the Cauchy problem, v is defined by this data uniquely everywhere in Ω. Thus, $f(k_0, n)$ is uniquely defined for all $n \in S^2$.

(2) Assume that there are two surfaces Γ_1 and Γ_2 such that the corresponding scattering amplitudes f_1 and f_2 are identical for $n \in S^2$. Then, by Rellich's lemma, $u_1 \equiv u_2$ for $|y| > R$, where R is the radius of a ball which contains Γ_1 and Γ_2. Here u_1 (u_2) solves (1) and (2) with $\Gamma = \Gamma_1$ (Γ_2). Thus, $u_1 = 0$ on Γ_2. If Γ_1 and Γ_2 do not have common points then f is the scattering amplitude simultaneously for a single reflecting convex body D_1 and for two such bodies D_1 and D_2. This leads to a contradiction. Namely, the scattering amplitude corresponding to a single reflecting convex body is a meromorphic function of complex k with no poles above some curve $\text{Im}k = -a\ln(1+|k|) - b$, $a > 0$, $b > 0$ (Lax and Phillips[1] and Ramm[2]); on the other hand, the scattering amplitude corresponding to two strictly convex bodies D_1 and D_2, $D_1 \cap D_2 = \emptyset$, has infinitely many poles on some line $\text{Im}k = -b$, $b > 0$.[3] This excludes the possibility $D_1 \cap D_2 = \emptyset$.

Suppose that $D_3 = D_1 \cap D_2 \neq \emptyset$. Since D_1 and D_2 are convex, D_3 is also convex. Thus f is the scattering amplitude for D_1 and D_3. Again one obtains a contradiction since the purely imaginary poles of the scattering amplitudes corresponding to D_3 and D_1, $D_1 \supset D_3$, cannot all be the same.[2,4] In particular, $N_3(b) < N_1(b)$ if $D_3 \subset D_1$, where $N_j(b)$ is the number of the purely imaginary poles $-ib_m$, $b_m > 0$, $b_m < b$, of the scattering amplitude corresponding to the reflecting domain D_j.

Corollary.—No solution $u \not\equiv 0$ of (1)–(3) can have a closed surface Γ_1 of zeros, $\Gamma_1 \cap \Gamma = \emptyset$.

This concludes the proof.

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