Localization of Waves in a Fluctuating Plasma

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(Received 5 July 1983)

We present the first application of localization theory to plasma physics: Density fluctuations induce exponential localization of longitudinal and transverse electron plasma waves, i.e., the eigenmodes have an amplitude decreasing exponentially for large distances without any dissipative mechanism in the plasma. This introduces a new mechanism for converting a convective instability into an absolute one. Localization should be observable in clear-cut experiments.

PACS numbers: 52.25.Gj, 52.35.Py, 71.50.+t

In this Letter we show the relevance of the concept of localization by disorder to the propagation of waves in plasmas with a fluctuating density. In contrast to the usual WKB picture, a fluctuating density can prevent the energy of a wave from propagating to infinity, and instead imply an exponential spatial decay of this energy without any dissipative mechanism. The normal modes are localized and the plasma becomes a resonant cavity without sharp boundaries. That can allow the plasma to switch from an amplifying to an oscillating state: A convective instability can become absolute. A simple plasma experiment is proposed to evidence our predictions. It also should allow the study of localization as a function of disorder in a more continuous way than in solid-state physics.

Localization was first discovered in connection with the metal-insulator transition in a crystal with randomly scattered impurities: When the eigenfunctions for energy near the Fermi level become localized, the metal turns into an insulator (for a general review on localization, see Thouless[1]). For one and probably also two spatial dimensions, localization is a property of all the eigenfunctions of the Schrödinger operator $-\Delta+\mathscr{V}(\vec{x})$ for almost all spatially homogeneous random potentials $\mathscr{V}(\vec{x})$, whatever the amplitude W of the randomness. The eigenfunctions are decreasing exponentially at infinity, and the inverse of the rate of the exponential decay is called the localization length, ξ, and depends both on the disorder W and on the eigenvalue $\mathscr{E}$ (the energy of the state). In a three-dimensional situation, for a given disorder of amplitude W, there exists a critical value $\mathscr{E}_c(W)$ of the energy $\mathscr{E}$ below which eigenfunctions are exponentially localized, and beyond which they are not. We emphasize that localization exists for almost all realizations of the medium and is not the consequence of a phase cancellation due to averaging over the various configurations of disorder. Localization is a stronger property than total reflection by a semi-infinite medium, considered for instance by Sulem, Frisch, and co-workers[2] for Schrödinger or Helmholtz equations. This appears in the fact that the Schrödinger operator with a random potential has a pure point spectrum, i.e., an eigenvalue spectrum,[3] whereas the absence of transmission is *a priori* compatible with a continuous spectrum. Localization contradicts the usual image of wave propagation in a random medium as a series of independent scatterings. It can be physically interpreted in the two following ways: (i) A wave with a given frequency interferes destructively with wavelets randomly scattered by the medium; the amount of scattered wavelets with random phases increases with the distance so that the global amplitude eventually vanishes. (ii) With probability 1 the wave meets large regions where the potential has a periodicity that implies gaps in the Floquet or Bloch analysis for the given energy.

Longitudinal or transverse electron plasma waves in an unmagnetized plasma (hereafter referred to as l or t waves) are described by an equation of the type

$$-\partial^2\psi/\partial(vt)^2=[-\Delta+\mathscr{V}(\vec{x},t)]\psi, \qquad (1)$$

where ψ is the electric field, and $\mathscr{V}(\vec{x},t)=\omega_p^2(\vec{x},t)/v^2$ with $v=c$ for t waves[4] and $v=\sqrt{3}v_T$ for l waves [in that second case $\vec{x}$ must be one dimensional for Eq. (1) to be correct[4,5]]. First consider the case where density is time independent. Then the time Fourier transform of Eq. (1) yields the time-independent Schrödinger equation

$$[-\Delta+\mathscr{V}(\vec{x})]\tilde{\psi}=\mathscr{E}\tilde{\psi}, \qquad (2)$$

where the energy $\mathscr{E}=\omega^2/v^2$; ω is the pulsation of the mode. If the plasma density is random, localization appears as stated above. Localization is not the exclusive property of the Schrödinger equation. Therefore, we can expect that other waves whose dispersion relation depends on the density can localize in a fluctuating plasma, provided that the dependence on the density is local (in particular, the density should not be involved in convolutions). This is likely to be the case for the equations $\nabla\cdot[-\Delta+\mathscr{V}(\vec{x})-\mathscr{E}]\tilde{\psi}=0$ and $\nabla\wedge\tilde{\psi}=\vec{0}$ that rule l waves in dimension 3. The relevance of localization for lower hybrid waves could strongly modify the image of propagation with scattering predicted by a WKB treatment.[6]

When the wave with pulsation ω is excited at $\vec{x}=\vec{0}$, a source term $\alpha\delta(\vec{x})$ must be added on the right-hand side of Eq. (2). As usual, the way to solve this new equation is to add to it a small imaginary term $i\epsilon$ corresponding to an artificial dissipation term and then find the solution $\tilde{\psi}$ of the equation, which is of course the Green's function G, before letting $\epsilon\rightarrow 0$ in the physical quantities computed with G. We note that

$$G(\vec{x})=-\{[-\Delta+\mathscr{V}(\vec{x})-\mathscr{E}-i\epsilon]^{-1}\alpha\delta\}(\vec{x}),$$

which, in the case of localized modes $\tilde{\psi}_j$ with eigenvalue $\mathscr{E}_j$, can be rewritten

$$G(\vec{x})=\sum_j\alpha\tilde{\psi}_j^*(\vec{x})\tilde{\psi}_j(\vec{0})/(\mathscr{E}_j-\mathscr{E}-i\epsilon).$$

For a given localized mode $\tilde{\psi}_j$, let its maximum value be at $\vec{x}_j$. If $\vec{x}_j$ does not coincide with $\vec{x}=\vec{0}$, the coefficient $\tilde{\psi}_j(\vec{0})$ which appears in the response of the system to a source term shows that the mode, if excited, is more weakly excited the larger $\|\vec{x}_j\|/\xi(\mathscr{E}_j)$ is, where $\xi(\mathscr{E}_j)$ is the localization length at eigenvalue $\mathscr{E}_j$ (this is reminiscent of the excitation of modes in a resonant cavity). Physically, there always is a finite frequency bandwidth for a source of waves, and hence there is an infinite number of excited modes: This is also true in the case of localized modes, since it follows from localization theory that the eigenvalues are then dense. The most strongly driven modes are those with a corresponding $\|\vec{x}_j\|$ of order less than or equal to $\xi(\mathscr{E}_j)$. Hence we predict an exponential spatial decay of l and t waves excited by an antenna in a fluctuating plasma, provided that the values of $\mathscr{E}$ corresponding to the frequency bandwidth lie in the domain of localized energies.

As yet no analytic formula is available that gives explicitly the localization length ξ for physically relevant random potentials and for all regimes of disorder. For one-dimensional systems, the localization length can be computed[7] for a small disorder W and scales like W^{-2}. For larger disorder, we must resort to numerical computations, made easy by the fact that, for one-dimensional systems, the localization length is the inverse of the Lyapunov exponent of Eq. (2) when rewritten as $d(\tilde{\psi},d\tilde{\psi}/dx)/dx=\mathscr{M}(x)(\tilde{\psi},d\tilde{\psi}/dx)$. This equation yields a matrix equation in the case of the simple random density $n(x)=n_0+\delta n(x)$, where n_0 is the average density, and where $\delta n(x)$ is constant on mutually decorrelated steps of length l_c and takes values between $-\Delta n$ and Δn with a uniform probability density. Let $\phi=l_c\,d\tilde{\psi}/dx$, $\phi_j=\phi(jl_c)$, $\psi_j=\tilde{\psi}(jl_c)$, $\overline{\omega}_p^2=n_0e^2/\epsilon_0 m$, $v=c$ for t waves and $v=\sqrt{3}v_T$ for l waves, $E=(\omega^2-\overline{\omega}_p^2)l_c^2/v^2$, and $W=(\overline{\omega}_p^2l_c^2/v^2)\Delta n/n$. One gets $(\psi_{j+1},\phi_{j+1})=M_j(\psi_j,\phi_j)$, where M_j is a 2×2 matrix that depends on $\delta n(jl_c)$. The Lyapunov exponent $\gamma=l_c/\xi$ depends on the parameters E and W, and is numerically computed. Figure 1 displays γ vs W/E for various values of E. Localization theory shows that ξ is not very sensitive to the model for $\delta n(x)$, and other random fluctuations with the same rms Δn and correlation length l_c should give similar ξ's. For instance consider an l wave with a wave number k when $\Delta n=0$, such that $k\lambda_D=0.1$ which corresponds to a negligible Landau damping. For a density fluctuation with $l_c/\lambda_D=10$ and $\Delta n/n=3\times10^{-2}$, one gets $\gamma=3\times10^{-2}$, i.e., $\xi\simeq5\lambda$ with $\lambda=2\pi/k$. This corresponds to $W=E=1$, that is, to a nontrivial localization ($W\leq E$). The condition for the WKB description of an l or t wave in the fluctuating plasma is $\Delta k/l_c<<k^2$, where Δk is the variation of k related to $\Delta n/n$. This condition is easily rewritten as $W<<2E^{3/2}$. It is weakly verified for the previous example though localization is fairly strong. Notice, however, that taking into account the complex turning points of Eq. (2) makes it possible to derive a WKB formula consistent with localization.[8]

For higher-dimensional systems, the computation of the localization length is much harder. One possibility would be to make a continuous analog of the scaling method used for discrete equations and based on the study of the sensitivity of eigenvalues of Eq. (2) in a box when changing boundary conditions.[9]

Till now, we only considered a static potential, $\mathscr{V}$, but, in a plasma a random density is also evolving with time. We are interested in two different situa-

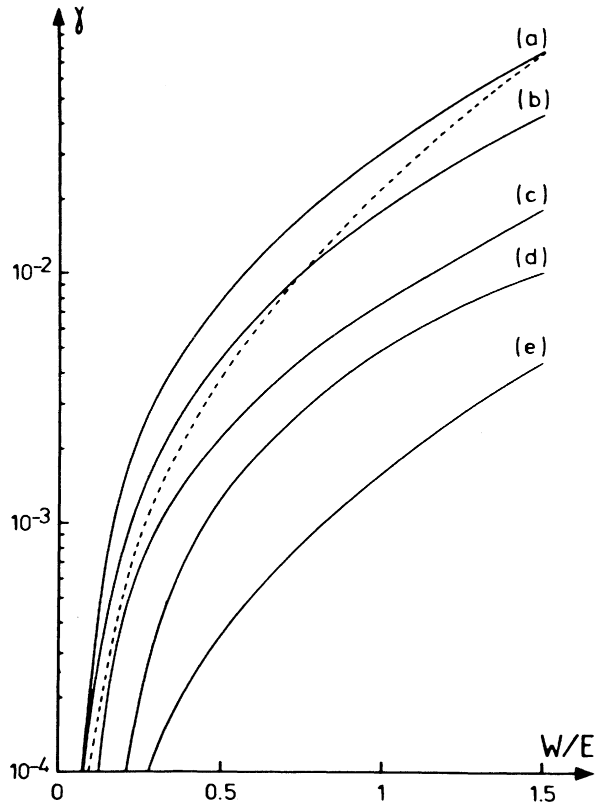

FIG. 1. Plot of $\gamma = l_c/\xi$ vs W/E for $E=1$ (curve a), 0.5 (curve b), 0.2 (curve c), 0.1 (curve d), and 0.05 (curve e); the dashed curve corresponds to $\gamma/2$ for $E=5$.

tions. In the first one, the random density is moving in the plasma with a constant velocity u much smaller than v: This is the case if, for instance, randomness is produced by a one-dimensional spectrum of ion-acoustic waves. The wave equations can then be rewritten in a frame moving with the density profile and this, together with an appropriate change of phase on ψ, leads directly to an Eq. (1) with appropriate $\mathscr{V}$ and $\mathscr{E}$: Previous conclusions about localization are hence applicable. Therefore, localization should be experimentally observable, for instance, for l waves in a magnetized plasma with ion-acoustic fluctuations.

The second case of interest is the one where the potential, or the density, $\mathscr{V}(\vec{x},t)$ is a random function both with respect to space and time, or is a random function in space varying with time without conserving a fixed profile, in opposition to the situation discussed above. The analysis of Schrödinger equations with time-varying potentials is much less developed than the one with static ones. The localization phenomenon, strictly speaking, does not exist any longer; localization is a subtle interference phenomenon and phase memory is lost when the potential changes. However, we can rely on some adiabatic treatment. Since the eigenvalues are a countable dense set in the spectrum, one cannot apply the usual adiabatic theory; but one can use the fact that wave functions for near energy levels are very separated in space. (Incidentally, this phenomenon is associated with the vanishing of the static conductivity in condensed-matter physics when the Fermi level lies among the energies of localized states.) We then expect that localization will still manifest its effects when the characteristic localization time is small with respect to the correlation time τ_c of $\mathscr{V}(x,t)$ or, in other words, if the energy can fill a cavity of length ξ during the correlation time, that is, if $\tau_c \gg \xi/w$, where w is the velocity of the energy of an incoming wave (it can be taken as the group velocity of the wave at $W=0$ if W is not too large).

In the case of l waves, the localization phenomenon must be discussed in the face of another phenomenon, namely, Landau damping: For a local excitation of a frequency (e.g., by an antenna in a plasma) it competes with localization for damping the wave amplitude of a driven mode with real frequency ω. In fact, values of ω close to $\overline{\omega}_p$ are well localized and experience little Landau damping, whereas the opposite occurs for large values of ω (typically $2\overline{\omega}_p$). We can therefore expect a crossover of these two regimes for some intermediate value of ω. Landau damping induces the temporal damping of a normal mode with a given frequency. If there is a gentle bump instability in the plasma, results for the WKB regime[10] show that the damping can change to a growth in a given range of frequencies. Thus localized modes can grow exponentially with time, i.e., *absolute instability* can set in; a better knowledge of the wave-number spectrum of localized modes has to be achieved for a full understanding of these wave-particle interaction effects.

It has been known for several years[11] that some absolute parametric instabilities which turned into convective ones by introduction of a density gradient could be made absolute again if random fluctuations were added to the gradient. This mechanism for turning convective instabilities into absolute ones seems very different from the one we have proposed above, since a constant density gradient from $-\infty$ to $+\infty$ destroys the localization created by the random fluctuations.[12]

Here are some conclusions and discussions: (i) The previous results show that the phenomenon of localization of waves by disorder should be experimentally observable in a plasma. This Letter proves it at least for Langmuir waves in a magnetized plasma, in the presence of ion-acoustic noise traveling in one direction. (ii) We suggest to study the localization-delocalization transition in three-dimensional plasmas. The disorder can then be easily varied in contrast to usual condensed-matter samples. But the first experimental check should be on one-dimensional localization; as a matter of fact, the first step could be the experimental observation of the effect of a coherent density fluctuation (for instance an ion-acoustic wave) on an l or t wave, such that $\mathscr{E}$ corresponds to a gap in the Mathieu equation. (iii) Modification of the gentle bump instability into an absolute one, due to localization, is one more phenomenon that could come about in the type-II solar-burst problem (notice that the existence of localized modes implies the existence of two reservoirs of counterstreaming plasmons that makes efficient the conversion of l waves into t waves at $2\omega_p$). (iv) The issue of the destabilization of localized modes motivates the study of the space Fourier transform of localized eigenfunctions, a new question raised by the application of localization to plasma physics.

Useful discussions with F. Delyon and G. Laval are acknowledged. The latter part of this work was supported by National Science Foundation Grant No. 26-1120-3562.

[(a)]One leave from Laboratoire de Physique des Milieux Ionisés, Ecole Polytechnique, F-91128 Palaiseau Cedex, France.

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