Quantum Fluctuations and the Lorenz Strange Attractor

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The full quantum mechanical master equation for the laser is used to obtain the effect of quantum fluctuations on the Lorenz strange attractor. For small quantum fluctuations the strange attractor survives with a different topology. For larger quantum fluctuations the attractor disappears and is replaced by limit cycles or fixed points, depending on the strength of the fluctuations.

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The Lorenz equations[1] developed for the Rayleigh-Bénard problem in fluid dynamics have been shown[2] to be isomorphic to the Maxwell-Bloch equations for a single-mode laser in the mean-field approximation.[3] It is then meaningful to inquire about the role of quantum fluctuations and the stability or otherwise of the strange attractor in their presence. Currently the impression is sometimes given that fluctuations may be adequately incorporated by adding Langevin Gaussian white-noise terms to the Lorenz equations. In a general context such a philosophy has been criticized by van Kampen[4] as being erroneous. Specifically for the Lorenz equations such an approach would only be correct if these were operator equations and the noise terms came multiplied by the operators of the Lorenz equation. Such a procedure is clearly far removed from that of adding simple Gaussian white-noise terms to *c*-number equations. We shall use the quantum statistical theory of the laser based on density matrices.

The master equation for a system of a dilute gas of a large number N of two-level atoms in a resonant cavity has the following form in the interaction picture[5]

$$\partial\rho/\partial t = (-i/\hbar)[H_I,\rho] + \Lambda_F\rho + \Lambda_A\rho, \tag{1}$$

where

$$H_I = ihg(a^\dagger R^- - R^+ a), \tag{2}$$

$$\Lambda_F\rho = \kappa([a,\rho a^\dagger] + [a\rho, a^\dagger]), \tag{3}$$

$$\Lambda_A\rho = \textstyle\sum_i\{(\gamma+\Gamma)([R_i^-,\rho R_i^+] + [R_i^-\rho, R_i^+]), \\ +\Gamma([R_i^+,\rho R_i^-] + [R_i^+\rho, R_i^-])\}, \tag{4}$$

$$R^{\pm} = \sum_{i=1}^{N} R_i^{\pm} e^{\pm i\vec{k}\cdot\vec{r}_i}. \tag{5}$$

Here ρ denotes the density matrix, g a coupling constant proportional to the dipole matrix element of an atom, κ, Γ, and γ are relaxation rates, $R_i^{\pm}$ are the pseudospin raising and lowering operators of the ith atom at $\vec{r}_i$, $\vec{k}$ is the wave vector, and a is the annihilation operator of the resonant mode of the cavity. The commutation relations of the operators are

$$[R_i^+, R_j^-] = 2R_{3i}\delta_{ij}, \tag{6}$$

$$[a,a^\dagger] = 1, \tag{7}$$

On introducing a pump term in the standard way[6] and on taking expectation values of a, R^-, R_3, and $e^{-ikr_j}R_j^- a^\dagger$ with the aid of Eq. (1), we find that the equation of motion for $\langle R^-\rangle$ has a linear term in $\langle aR_3\rangle$ and the corresponding equation for $\langle e^{-ikr_j}R_j^- a^\dagger\rangle$ has a term linear in $\langle a^\dagger aR_3\rangle$ and also a term proportional to

$$\sum_{\substack{i\\(\neq j)}} \langle R_j^- e^{-ikr_j} R_i^+ e^{ikr_i}\rangle .$$

As is usual in laser theories, a dilute gas of atoms is assumed. This implies that the last correlation function can be factorized. However, the system of

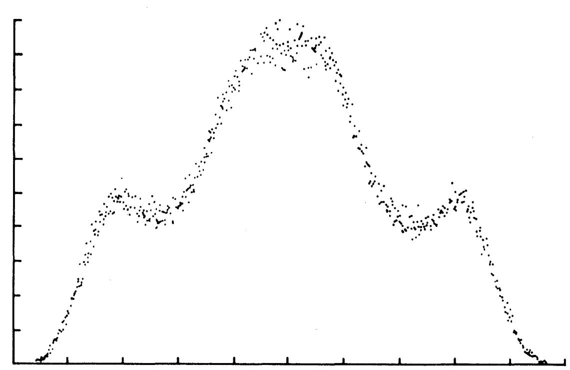

FIG. 1. Probability distribution of X for $\epsilon = 0$.

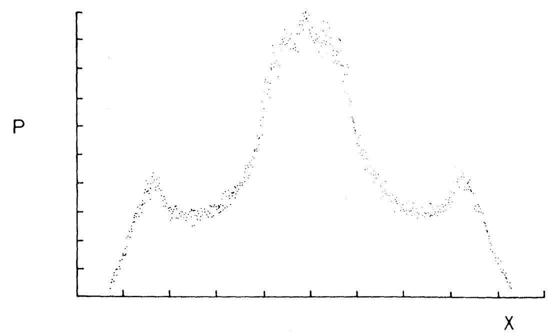

FIG. 2. Probability distribution of X for $\epsilon = 10^{-4}$.

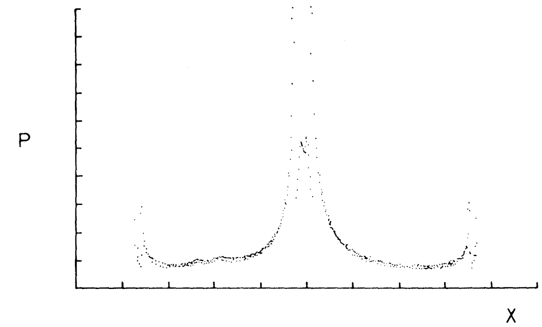

FIG. 3. Probability distribution of X for $\epsilon = 10^{-3}$.

equations is still not closed. We now adopt the semiquantum approximation used quite extensively for the laser.[7,8] In our problem this implies that

$$\langle aR_3\rangle \simeq \langle a\rangle\langle R_3\rangle,$$

and

$$\langle a^\dagger aR_{3j}\rangle \simeq \langle a^\dagger a\rangle\langle R_{3j}\rangle.$$

In this way phase correlations are preserved (i.e., we do not decorrelate $\langle a^\dagger R^-\rangle$ for example). Mandel[7,8] has shown that such approximations are better than the semiclassical approximation for the laser (both above and below threshold) since spontaneous emission is included in this approximation. An equation for $\langle a^\dagger a\rangle$ is then necessary to close the set; the following equations are then found:

$$\dot X = \sigma(Y - X), \tag{8}$$

$$\dot Y = rX - ZX - Y, \tag{9}$$

$$\dot Z = XY - bZ + b(r-1)U, \tag{10}$$

$$\dot U = V(r-Z) - (\sigma+1)U + 4\sigma\epsilon(R-Z), \tag{11}$$

$$\dot V = 2\sigma(U - V), \tag{12}$$

where

$$X \propto \langle a\rangle,$$

$$Y \propto \langle R^-\rangle,$$

$$r - Z \propto R_3,$$

$$U \propto \langle a^\dagger R^-\rangle - \langle a^\dagger\rangle\langle R^-\rangle,$$

$$V \propto \langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle.$$

Here 2ϵ is $g^2/\kappa(\gamma+2\Gamma)(r-1)$ and R is $[g^2N/\kappa(\gamma+2\Gamma)+r]$. The constants r, σ, and b have the same meanings as in the standard Lorenz model.

It is difficult, as in the usual Lorenz system, to obtain much information about the system of Eqs. (8)–(12) analytically. However, it is easy to find the fixed points of the flow. These are

$$X = \pm(1-2\epsilon)^{1/2}[b(r-1)]^{1/2}, \tag{13}$$

$$Y = \pm(1-2\epsilon)^{1/2}[b(r-1)]^{1/2}, \tag{14}$$

$$Z = r - 1, \tag{15}$$

$$U = V = 2\epsilon. \tag{16}$$

For small ϵ these are very near the nontrivial

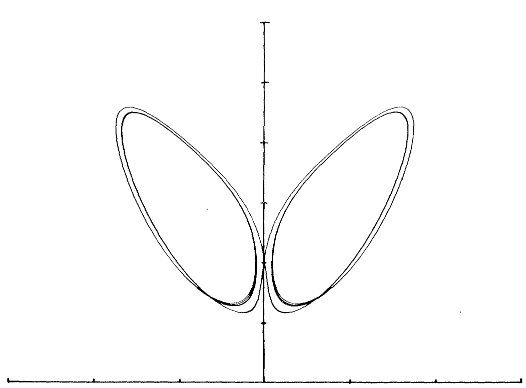

FIG. 4. Phase space plot of Z vs X for $\epsilon = 10^{-3}$.

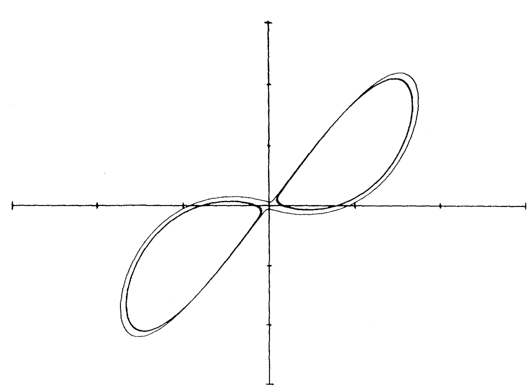

FIG. 5. Phase space plot of Y vs X for $\epsilon = 10^{-3}$.

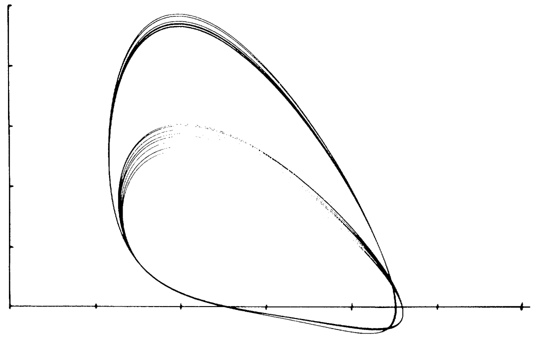

FIG. 6. Phase space plot of U vs Z for $\epsilon = 10^{-3}$.

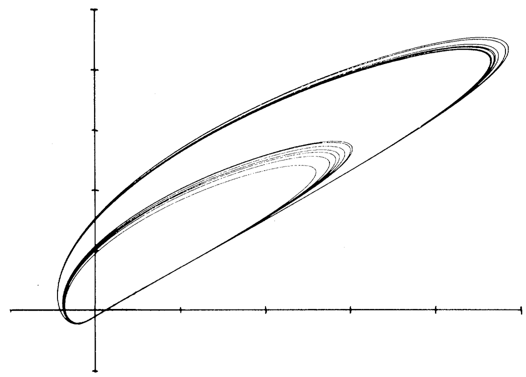

FIG. 7. Phase space plot of V vs U for $\epsilon = 10^{-3}$.

fixed points of the Lorenz model (on projection onto the X, Y, Z plane) and unlike the Lorenz system there is no trivial fixed point (but it is only above the lasing transition that the model is valid). When we remember that g is proportional to $V^{-1/2}$, where V is the active volume of the cavity, in the thermodynamic limit ϵ will decrease as $1/N$ where N is the number of atoms. For very small ϵ it is possible to repeat the analysis of the usual Lorenz case and find that the threshold for the onset of chaos is modified by $O(\epsilon)$. In fact a stability analysis can be given even when ϵ is not infinitesimal. The evaluation of the eigenvalues of the stability matrix involves finding the roots of a quintic which is done numerically. For $\epsilon = 0$ and 10^{-4} there is a complex conjugate pair of eigenvalues with positive real part in the chaotic region. For $\epsilon = 10^{-3}$, however, all eigenvalues have negative real parts when $r = 25$ but for sufficiently large r (e.g., $r = 28$) a complex-conjugate pair of eigenvalues attains a positive real part. For $\epsilon = 10^{-2}$ and larger all fixed points are stable for parameter values which would correspond to chaos in the ordinary Lorenz model. In summary we find that as the ratio ϵ/r increases chaos is suppressed.

Figures 1–7 show different two-dimensional cross sections of typical modifications to the Lorenz attractor due to quantum fluctuations. For $\epsilon = 10^{-4}$ although the Z-X plot is similar to the plot in the conventional Lorenz attractor, the U-V, U-Z, and V-Z plots are topologically quite different and are somewhat similar to corresponding plots for $\epsilon = 10^{-3}$. For the latter value of ϵ the Z-X and Y-X plots are to a fine resolution periodic but the V-U and U-Z plots are quite aperiodic. The graphs for the probability distributions of X reflect the structure of these attractors. On comparing the distributions for $\epsilon = 0$ and $\epsilon = 10^{-4}$ we find that the central and side peaks have much sharper cusps when there are fluctuations. For $\epsilon = 10^{-3}$ the distribution is quite different and confirms the presence of a strong pseudoperiodic element in the dynamics.

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