General Correlation Identity for Parametric Processes

Robert Graham
Fachbereich Physik, Universität Essen, D-4300 Essen, West Germany
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A new connection between autocorrelation and cross correlation functions of the intensities of signal and idler in parametric oscillators is derived, with the same origin and generality as the Manley-Rowe relations and with interesting experimental implications in the quantum domain.

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Parametric oscillators are extremely useful tools with broad applications in many branches of physics and technology.[1] Their basic principle, the simultaneous excitation of several coupled modes of different frequencies by a periodic variation of their coupling parameter at the sum frequency, has been realized in all frequency domains up to the optical domain.

Because of their common underlying principle all parametric oscillators share certain fundamental properties. One property of particular generality is the relation discovered by Manley and Rowe.[2] It states that in the steady state the power extracted from or dissipated in any of the simultaneously excited modes is proportional to their respective frequency, i.e., if P_i is the power of the mode with frequency ω_i then in the steady state

$$\frac{P_1}{\omega_1}=\frac{P_2}{\omega_2}=\ldots=\frac{P_n}{\omega_n}. \qquad (1)$$

In this Letter I wish to prove another fundamental relation which holds with the same generality as Eq. (1) and which has, in fact, a common origin. It is a relation between the intensity cross-correlation function and the intensity autocorrelation functions of any pair of the simultaneously excited modes (called signal and idler, in the following). Since this relation is of particular interest in the optical domain I will present a quantum statistical derivation, which also proves (1) in the quantum domain. Despite its great generality the relation that is proven here seems to have escaped notice, so far. A special case had been obtained earlier from an approximate analysis of a special model,[3] but its general significance was not recognized. The same special case of the relation is also implicit in the results of a recent paper[4] in which a somewhat similar analysis was presented, but the relation went unnoticed there.

The derivation will be based on an interaction Hamiltonian of the general form[5]

$$H_{\text{int},p}=i\hbar g(b_p b_1^\dagger b_2^\dagger\cdots b_n^\dagger-\text{H.c.}), \qquad (2)$$

where g is the coupling constant, b_i and $b_i^\dagger$ are the annihilation and creation operators of the mode with frequency ω_i, and b_p and $b_p^\dagger$ are the corresponding operators of the mode at the pump frequency ω_p. I assume at least approximate resonance,

$$|\Delta\omega|=\left|\omega_p-\sum_{i=1}^{n}\omega_i\right|\ll\omega_k,\quad k=1,2,\ldots n,$$

so that nonresonant interaction terms are negligible. In addition to the interaction Hamiltonian there is a free-field Hamiltonian H_F of the usual form. The arbitrary external excitation of the pump mode is represented by an explicitly time-dependent Hamiltonian involving $b_p, b_p^\dagger$ only, and other operators describing the pumping light source, which are not written out explicitly:

$$H_p=H_p(b_p,b_p^\dagger,t). \qquad (3)$$

For arbitrary statistical excitation of the pump, $H_p(t)$ is an arbitrary stochastic (operator-valued) process in time. For multimode excitation, Eqs. (2) and (3) are to be summed over p.

The extraction of power from all modes is represented by their coupling to heat baths in a standard way.[5] For simplicity of presentation I neglect here finite-temperature effects in these heat baths (which is permitted at optical frequencies, but not at much lower frequencies), but the relation can easily be generalized to include such effects.

The quantum statistical description of a parametric oscillator is then provided by a master equation for its density operator ρ, which takes the form[5]

$$\partial\rho/\partial t=(i/\hbar)[\rho,H]+\sum_p L_p\rho+\sum_i L_i\rho, \qquad (4)$$

where $H=H_F+\sum_p(H_{\text{int},p}+H_p)$ and L_p and L_i describe dissipation in all modes and are of the form

$$L_\lambda \rho = \kappa_\lambda([b_\lambda \rho, b_\lambda^\dagger] + [b_\lambda, \rho b_\lambda^\dagger]), \quad \lambda = i, p. \tag{5}$$

In the following, I only use one basic property of H, namely the existence of the conserved quantities

$$\Delta_{ij} = b_i^\dagger b_i - b_j^\dagger b_j; \quad [H, \Delta_{ij}] = 0. \tag{6}$$

Because of the symmetry between all excited modes it is sufficient to consider Δ_{12} in the following.

If an oscillator is pumped from the vacuum state of signal and idler and losses in these modes are negligible, the state space is given by the subspace with $\Delta_{12} = 0$ and the intensities of signal and idler are maximally correlated at all times:

$$\langle b_1^\dagger b_1 \rangle = \langle b_2^\dagger b_2 \rangle, \tag{7}$$

$$\langle b_1^\dagger b_2^\dagger b_2 b_1 \rangle = \langle b_1^\dagger b_1^\dagger b_1 b_1 \rangle + \langle b_1^\dagger b_1 \rangle. \tag{8}$$

Equation (7) is the lossless version of (1) and follows from $\Delta_{12} = 0$. Equation (8) follows from $\Delta_{12}^2 = 0$ if use is made of the complete symmetry of the Hamiltonian with respect to signal and idler and if we put all correlation functions into the normally ordered form which is measured in photocount experiments.[6] Equation (8) is the lossless special case of the general relation which I want to establish here. The intensity cross-correlation function, according to Eq. (8), takes the maximum value which is permitted by a general Schwartz inequality.[4]

For an oscillator with finite losses the difference Δ_{12} is no longer conserved. However, general relations of the kind (7), (8) may still be established in the steady state, where the averages $\mathrm{Tr}\rho\Delta_{12}$ and $\mathrm{Tr}\rho(\Delta_{12})^2$ become time independent. Using the master equation (4), (5) with $[H, \Delta_{12}] = 0$, we obtain from $\mathrm{Tr}\dot{\rho}\Delta_{12} = 0$ the simple relation

$$\kappa_1 \langle b_1^\dagger b_1 \rangle = \kappa_2 \langle b_2^\dagger b_2 \rangle. \tag{9}$$

Since the total power P_i extracted from mode i is given by $P_i = 2\kappa_i \hbar\omega_i \langle b_i^\dagger b_i \rangle$ Eq. (9) proves Eq. (1). In the same way we obtain from $\mathrm{Tr}\dot{\rho}\Delta_{12}^2 = 0$ with $[H, \Delta_{12}^2] = 0$ and some simple rearrangements,

$$\langle b_1^\dagger b_2^\dagger b_2 b_1 \rangle = \frac{\kappa_1}{\kappa_1 + \kappa_2} \langle b_1^\dagger b_1^\dagger b_1 b_1 \rangle + \frac{\kappa_2}{\kappa_1 + \kappa_2} \langle b_2^\dagger b_2^\dagger b_2 b_2 \rangle + \frac{\kappa_1}{\kappa_1 + \kappa_2} \langle b_1^\dagger b_1 \rangle. \tag{10}$$

It is clear that identities for higher-order correlation functions could be similarly derived with $[H, (\Delta_{12})^n] = 0$.

I note that Eqs. (9) and (10) can only be established for the steady state and are therefore similar in type but fundamentally different from Eqs. (7) and (8) which apply to the lossless time-dependent state. In fact, neither Eq. (7) nor (8) is obtained from Eqs. (9) and (10) in the singular limit $\kappa_1 \to 0$, $\kappa_2 \to 0$, since the generally time-dependent state without any losses is not the limit of a steady state with small losses.

However, if no *internal* losses occur in the parametric system, the κ_i in Eq. (10) can be identified with the rate constant of radiation from mode i and Eq. (10) may be rewritten as a relation between photocount rates

$$R_i = 2\eta_i \kappa_i \langle b_i^\dagger b_i \rangle,$$

$$R_{ij} = 2\eta_i \eta_j (\kappa_i + \kappa_j) \langle b_i^\dagger b_j^\dagger b_j b_i \rangle.$$

Here η_i is the quantum efficiency of the counter for mode i, R_i is the average counting rate for mode i, and R_{ij} is the coincidence rate for simultaneous photocounts of mode i and mode j. R_{ii} is measured in mode i by using a beam splitter and two identical counters. Equation (10) then states

$$R_{12} = \tfrac{1}{2}[(\eta_2/\eta_1)R_{11} + (\eta_1/\eta_2)R_{22}] + \eta_2 R_1.$$

For ideal quantum efficiencies $\eta_i = 1$, this relation just repeats Eq. (8) for counting rates. If there are finite internal losses in the parametric system, the κ_i in Eq. (10) are different from and larger than the rate constants appearing in the definition of the counting rates R_i, R_{ij}.

While the fundamental significance of the Manley-Rowe relations (1) for parametric processes has long been appreciated, the equally general relation (10) between the intensity cross-correlation function of signal and idler and their autocorrelation functions seems to be new. For the case of equal κ_i Eq. (10) reduces to

$$\langle b_1^\dagger b_2^\dagger b_2 b_1 \rangle = \langle b_1^\dagger b_1^\dagger b_1 b_1 \rangle + \tfrac{1}{2} \langle b_1^\dagger b_1 \rangle.$$

In this special case, the relation has been obtained in a previous paper[3] as the result of an approximate analysis under much more special assumptions; it is also implied by the results of a more recent paper[4] obtained under very similar special assumptions.

I now discuss the general quantum mechanical significance of the present result and present a comparison with the available experimental work. In the completely lossless case the joint quantum state of signal and idler for $\Delta_{12}=0$ is described by a pure quantum state of the form $|\psi\rangle=\sum_{n_2}c(n_2)\times|n_2,n_2\rangle$, where $|n_1,n_2\rangle$ is the number state with n_1 and n_2 quanta in the signal and idler, respectively. The measurement of the quantum number in the idler mode determines the quantum number in the signal mode completely and instantaneously. This complete quantum correlation is expressed by Eq. (8). Quantum correlations of this type have been first discussed by Einstein, Rosen, and Podolsky[7] and the joint measurement of all the correlation functions in Eq. (8) constitutes an experiment of the type proposed by them. It would allow one to distinguish experimentally between quantum statistics and classical statistics in a pure quantum state. While such experiments have so far only been conceived in lossless conservative systems (mainly for correlations between spins), the relation (10) suggests that we consider a similar experiment for a quantum optical system in a dissipative steady state. An experiment of this kind would test an intrinsical quantum mechanical property of a dissipative system and would therefore be of fundamental importance. The predictions of Eq. (10) for such an experiment in the case of symmetrical losses are as follows: (i) The intensity cross-correlation function in the steady state is reduced below its theoretical maximum given by Eq. (8) because of dissipation, but (ii) it is still larger than the classically allowed maximum value[8] given by $\langle b_1{}^\dagger b_2{}^\dagger \times b_2 b_1\rangle=\langle b_1{}^\dagger b_1{}^\dagger b_1 b_1\rangle$. In fact, the cross-correlation function is predicted to lie halfway between both cases. (iii) The correlation identity (10) and the Manley-Rowe relations (9) should hold completely independently from the way in which the parametric oscillator is pumped; in particular, an incoherent multimode pump may be used.

Measurements of the intensity cross correlation between signal and idler in parametric fluorescence have been carried out.[9] In that case modes satisfying frequency matching $\omega_p=\omega_1+\omega_2$ and wave-number matching $k_p=k_1+k_2$ are pairwise coupled by the Hamiltonian (2), and the system is driven far below the threshold of self-sustained oscillation. In that region the field statistics of signal and idler is known to be Gaussian and their phases are random[5] but with strong correlations between signal and idler, which are completely determined by Eq. (10). Using the Gaussian statistics on the right-hand side of Eq. (10), we obtain with Eq. (9)

$$\langle b_1{}^\dagger b_2{}^\dagger b_2 b_1\rangle = 2\langle b_1{}^\dagger b_1\rangle\langle b_2{}^\dagger b_2\rangle+\frac{\kappa_1}{\kappa_1+\kappa_2}\langle b_1{}^\dagger b_1\rangle. \tag{11}$$

Using the Gaussian statistics also on the left-hand side of Eq. (10) and noting that $\langle b_1{}^\dagger b_2\rangle$ vanishes in the steady state because of the randomness of the phase difference between signal and idler, we obtain

$$|\langle b_1 b_2\rangle|^2=\langle b_1{}^\dagger b_1\rangle\langle b_2{}^\dagger b_2\rangle+\frac{\kappa_1}{\kappa_1+\kappa_2}\langle b_1{}^\dagger b_1\rangle. \tag{12}$$

The maximum quantum correlation allowed by the Schwartz inequality is given by

$$|\langle b_1 b_2\rangle|^2=\langle b_1{}^\dagger b_1\rangle\langle b_2{}^\dagger b_2\rangle+\langle b_1{}^\dagger b_1\rangle. \tag{13}$$

It is obtained in the completely lossless case from Eq. (8). In parametric fluorescence, the average quantum number in signal and idler is much smaller than 1 (it can be estimated to be of the order of 10^{-7} in the experiments of Ref. 9) and the first term on the right-hand side of Eq. (11) is completely negligible. The experiment therefore operates in the extreme quantum limit of Eq. (10), where the second term dominates the classical first term. Under the assumption that the internal losses in the experiment are negligible, Eq. (11) may be rewritten for counting rates and then reads $R_{12}=\eta_2 R_1$. This relation was found to be satisfied in Ref. 9 within the estimated experimental errors. To within these uncertainties, the experiment of Ref. 9 therefore establishes nonclassical correlations for signal and idler in parametric fluorescence, very far above the classically permitted maximum value, given by the first term of Eq. (10). Close to the classical limit the first term on the right-hand side of Eq. (11) or Eq. (10) is dominant. The transition between the quantum regime and the classical regime is described by the full relation (10). It could be checked in an experiment similar to that of Ref. 9 if a parametric oscillator is observed in a sequence of states from below threshold to high above threshold.

In summary, I have derived the general relation (10) between the autocorrelation functions and the cross-correlation function of the intensities of signal and idler in the steady state of a parametric oscillator. As was made clear by its derivation this relation applies regardless of the details of the pumping process.

It reflects the strong correlations of signal and idler due to their joint generation by a single quantum process. In the case without any losses, these correlations are so strong that the measurement of the quantum number of mode 1 determines the quantum number of mode 2 completely and instantaneously, providing an example of the Einstein-Podolsky-Rosen experiment. I have shown that a corresponding result holds for steady-state photon counting rates, if there are no internal losses in the parametric system beyond those due to the escape of the photons. If there are additional internal losses, relation (10) is still applicable and describes a cross correlation whose strength is reduced by dissipation. I therefore obtain the possibility for an experiment of the Einstein-Podolsky-Rosen type in a dissipative steady state. I have discussed explicitly the application of the results to experiments on parametric fluorescence in the steady state.

As a final remark it may be worthwhile to point out that the whole discussion applies without change also to photon correlations in nondegenerate two-photon lasers, which have recently been realized.[10]

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