Operational Approach to Phase-Space Measurements in Quantum Mechanics

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An operational formula is derived for a positive phase-space distribution function in quantum mechanics. It is shown that this expression, which is based on a realistic detection mechanism, has many attractive features. The possibility of an application of such a probability distribution to a nonlinear wave mechanics is also briefly mentioned.

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The idea of a phase-space probability distribution function in quantum mechanics has attracted a lot of interest since the pioneering work of Wigner[1] in which the relation

$$W_\psi(q,p) = \int \frac{dx}{2\pi\hbar} \psi^*\left(q+\frac{x}{2}\right)\psi\left(q-\frac{x}{2}\right)\exp\left(\frac{ipx}{\hbar}\right) \quad (1)$$

between the wave function ψ and a phase-space function W_ψ was proposed. It was shown later that this very simple relation is in fact unique if some general conditions are satisfied.[2,3] It was clear to Wigner from the beginning that W_ψ cannot really be interpreted as the phase-space probability distribution function because it may take negative values. In fact, Wigner has shown later that general conditions leading to unique definition of W_ψ are incompatible with $W_\psi \geq 0$ for all p and q.[2]

In order to overcome this apparent difficulty, various rather artificial smoothing procedures[4] of the Wigner function or different definitions[5] of the phase-space distributions have been proposed. A different approach based on the concept of the so-called fuzzy space[6] or the Menger-Wald statistical metric space has also been advocated in order to maintain a positive definite statistical phase-space description of quantum mechanics.

The main weakness of most of these approaches is how little attention they pay to the actual relation of such *ad hoc* procedures to realistic quantum mechanical measurements. In fact the very question why W_ψ should be a measurable probability distribution function in phase space at all has been completely missing in most of these discussions.

Following the opinion expressed by Lamb[7] that, "In discussion of the measurement of some dynamical variable of a physical system I want to know exactly what apparatus is necessary for the task and how to use it, at least in principle," there is little justification for the physical meaning of any of these phase-space distributions. If the main result of these different approaches to the definition of a phase-space distribution is only mathematical, i.e., if the introduced functions themselves are not regarded as measurable objects but only as a means of calculating measurable quantities, then the most fundamental question remains open: Is it possible to define a realistic phase-space function that can be recorded in the laboratory?

It is the purpose of this Letter to give an operational physical definition of a quantum mechanical phase space. In the spirit of Lamb's remark, I derive a positive definite quantum probability distribution $P(q,p)$ which is directly connected to a realistic measurement. As will be seen clearly later on, the fundamental feature of the operational $P(q,p)$ is that, in addition to the particle to be measured and its detector, it requires for its definition a device acting as a filter. This filter is needed, as a matter of principle, in order to resolve the current position and momentum of the investigated system.

The idea that in any realistic measurement a detector and a filtering device are required is not really quantum mechanical in nature. For spectral measurements in optics filtering devices such as the Fabry-Perot interferometer, for example, have been known for years.[8] What really is new and what has been realized for time and frequency measurements in optics and acoustics is a necessary influence of the filter property on any resolved observation of the phase-space dynamics.[9] All earlier mathematical attempts[10] to define a temporal evolution of a spectrum without the filtering mechanism resemble or in many cases are identical to various smoothing procedures of the quantum mechanical Wigner function. How important the filtering mechanism is for a proper definition of a quantum phase-space measurement will be seen below.

Now I shall propose an experimental setup in order to exhibit the role of the filtering mechanism in the derivation of an operational definition of a phase-space distribution function. This setup may not be the most convenient for practical measurements but has all the characteristics that permit us to perform an idealized but fairly general calculation, which I confirm at the end of this Letter from a much more formal mathematical argument.

For a possible scheme of a one-dimensional position and momentum measurement of a charged particle, I propose to use a pulsed interaction (laser pulses, for example)[11] with potential $U_q(x)$ centered around a detected position q and produced in $\delta(t-t_0)$-like form at time t_0. The wave function of the moving particle with velocity v couples to such a filter via the standard interaction potential $V_q(x,t)=U_q(x)\delta(t-t_0)$. Changing the parameter q of the interaction potential, we can scan all x. The result of the interaction then gives us information on the position of the particle at time t_0. By analogy to the detection mechanism in optics, this interaction potential plays the role of the filtering device which can be "tuned" by simply changing q. This filter scatters the wave function of the measured particle. In the Born approximation, the scattered wave function of the moving particle is given by

$$\phi_s(x,t)=\int dt' \int dx'\, K_0(x,t;x't')\, V_q(x',t')\phi(x',t'), \tag{2}$$

where K_0 is the free Schrödinger propagator whch in the far zone from the interaction region, along a straight line determined by the velocity of the particle $x=vt$, has the following asymptotic form[12]:

$$K_0(vt,t;x',t') \underset{t\to\infty}{\to} \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left(\frac{imv^2t}{2\hbar}\right)\exp\left(\frac{imv^2t'}{2\hbar}\right)\exp\left(-\frac{imvx'}{\hbar}\right). \tag{3}$$

The square of the modulus of the scattered wave measured by a detector located far from the interaction region contains now measurable information about the position q and momentum $p=mv$ of the detected particle ϕ at time t_0. The interaction potential $U_q(x)$, which for example in the case of laser pulse scattering is simply an electric field envelope, can be easily associated with an experimental quantum mechanical state of the filtering device denoted for notational reasons by $\psi^*(x+q)$. Following the Feynman approach,[13] we can write this squared modulus as

$$\left|\int dx'\, K_{\rm exp}(x,t;x',t_0)\phi(x',t_0)\right|^2$$

where $K_{\rm exp}=K_0V_q$ is the propagator for going through our experimental filtering device. This propagator is proportional to $K_0\psi^*(x+q)$, i.e., to a reference wave function of the filter.

From Eqs. (2) and (3) we obtain the following two-parameter expression for the squared modulus of the scattered wave that I propose to call the operational phase-space probability distribution:

$$P(q,p)=(2\pi\hbar)^{-1}\left|\int dx \exp(-ipx/\hbar)\psi^*(x+q)\phi(x)\right|^2. \tag{4}$$

The normalization coefficient in this expression is chosen in such a way that

$$\int dq \int dp\, P(q,p)=\langle\psi|\psi\rangle\langle\phi|\phi\rangle=1 \tag{5}$$

if the wave functions of the filter and of the measured object are normalized. The expression (4) is a central result of this paper. Except for the Planck-cell normalization, mathematically Eq. (4) is completely equivalent to an operational definition of a time-dependent spectrum of light.[9] As in the case of optical measurements, it involves both the filter and the detected object in its definition. In order to make a closer contact with the phase-space dynamics, I will show some remarkable properties of this quantum mechanical distribution function. The marginal average of $P(q,p)$ in q space has the following form:

$$P(q)=\int dp\, P(q,p)=\int dx\, \phi^*(x)\phi(x)\psi^*(x+q)\psi(x+q). \tag{6}$$

This equation leads to the following phase-space expectation value of q:

$$\langle q\rangle_P=\int dq\, qP(q)=\langle\hat{x}\rangle_\psi-\langle\hat{x}\rangle_\phi \tag{7}$$

with $\langle\hat{x}\rangle_\psi=\int dx\,\psi^*(x)x\psi(x)$. Formula (7) shows that $\langle q\rangle_P$ measures the relative position of the detected state ϕ with respect to a reference given by the position of the filter ψ. This result is in full agreement with the way in which we have constructed the phase-space probabiltiy distribution function from a detection mechanism with a reference filter device. The phase-space second moment of q has the following form:

$$\langle q^2\rangle_P = \int dq\, q^2 P(q) = \langle \hat{x}^2\rangle_\psi - 2\langle \hat{x}\rangle_\psi \langle \hat{x}\rangle_\phi + \langle \hat{x}^2\rangle_\phi, \tag{8}$$

i.e., as in the case of $\langle q\rangle_P$, the fluctuations of both the measuring and detected wave functions contribute.

For noncommuting observables the statistical average in phase space is more complicated and, for example, we have

$$\langle qp\rangle P = \int dq \int dp\, qpP(qp) = \langle qp\rangle_\psi - \langle q\rangle_\phi \langle p\rangle_\psi - \langle q\rangle_\psi \langle p\rangle_\phi + \langle qp\rangle_\phi, \tag{9}$$

where $\langle qp\rangle_\psi = \frac{1}{2}\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle_\psi$, i.e., a Weyl ordering of quantum operators is obtained. The appearance of the Weyl ordering in Eq. (8) and the easy-to-obtain general pattern for all higher phase-space moments of q and p result from the following connection of $P(q,p)$ with the Wigner function (1):

$$P(q,p) = \int dx \int dp'\, W_\psi(q+x, p+p')\, W_\phi(x,p'), \tag{10}$$

i.e., the operational phase-space distribution function is given by an overlap of the *detected* and of the *filtering* Wigner functions. Equation (10) can be simply obtained by combining the definition of the Wigner function with the operational phase-space probability distribution given by Eq (4). Equation (10) answers also the fundamental question of the proper relation between the Wigner distribution function and a realistic phase-space measurement in quantum mechanics. In fact, this relation is universal. In optics and acoustics Eq. (4) or (10) is very closely related to the time-dependent spectrum resolved by the frequency filtering device.[9,14]

We can obtain now the operational phase-space distribution probability (4) from a more formal argument.[15] The probability of finding a state $|\psi\rangle$ in another state described by the density matrix $\hat{\rho}$ is equal to $\mathrm{Tr}(\hat{\rho}\hat{P})$ where $\hat{P} = |\psi\rangle\langle\psi|$. We can regard the projection operator $\hat{P}$ as the filtering device. In order to compare states in phase space, we need to "shift" one with respect to the other by amounts q and p, respectively. In the q coordinate, this shift can be done by the operator $\exp(iq\hat{p})$ where $\hat{p}$ is the space translation generator. In the p space, the operator $\exp(ip\hat{x})$ can be used because $\hat{x}$ is the Galilean boost generator. Combining the two operations in the unitary operator $\hat{U}(q,p)$ we introduce

$$P(q,p) = (2\pi\hbar)^{-1}\mathrm{Tr}[\hat{\rho}\hat{U}^{-1}(qp)\hat{P}\hat{U}(qp)]. \tag{11}$$

For a pure state, i.e., if $\hat{\rho} = |\phi\rangle\langle\phi|$, Eq. (11) reduces to the formal (4). Equation (11) can be easily generalized to other unitary transformations involving, for example, rotations or Poincaré transformations giving in such a way probability distributions associated with different "conjugated variables."

From Eq. (4) we see that

$$P(0,0) = (2\pi\hbar)^{-1}|\langle\psi|\phi\rangle|^2, \tag{12}$$

i.e., a very close relation between a phase-space distribution and the Hilbert-space scalar product holds. In an attempt to interpret the Hilbert-space scalar product in terms of a relation or a propensity between two or more distinct physical systems, objects close to Eqs. (10) and (12) have been discussed recently.[16,17] Here we have given a full operational justification for Eq. (10), deriving it from a realistic dynamical filtering process [see Eqs. (2) and (4)]. From this derivation it is clear that ψ should be attributed to the state of the filtering device.

At this point, we see a possible flexibility of the quantity $P(q,p)$ for theories which propose a nonlinear generalization of the Schrödinger equation.[18] For such nonlinear equations, the standard Hilbert-space interpretation of quantum mechanics does not hold any more. Nevertheless, for such theories the concept of a phase-space probability distribution can have a very well defined operational meaning.

Let us illustrate this point for a hypothetical detector based on our light-pulse filter which has its own internal dynamics that can be described in the scattering process only by the second-order Born approximation. Following calculations similar to the one leading to Eq. (4) we obtain for our nonlinear filter

$$P(q,p) = \int dx \int dp'\, W_\psi^{(2)}(q+x, p+p')\, W_\phi(x,p'), \tag{13}$$

where

$$W_\psi^{(2)}(q,p) = \int (dx/2\pi\hbar)\, \psi^{*2}(q+\tfrac{1}{2}x)\, \psi^2(q-\tfrac{1}{2}x) \exp(ipx/\hbar) \tag{14}$$

is a generalized quartic Wigner function. From Eq. (13) we obtain

$$P(0,0)=(2\pi\hbar)^{-1}\left|\int dx\,\psi^{*2}(x)\phi(x)\right|^2$$

and an obvious non-Hilbert structure between the measured state ϕ and the detector state ψ does emerge.

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