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New Integrable Hamiltonians with Transcendental Invariants

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This paper presents three two-dimensional integrable Hamiltonians, whose second invariant I_2 is rational or transcendental in momenta. A third invariant I_3 (canonically conjugate to I_2) is also found for them, and the motion of the particle explicitly solved. The original Painlevé test does not work very well for these models; however, expansions with four resonances are found in each case.

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Recently there has been much effort given for finding integrable Hamiltonian models. To find such models one has to solve the partial differential equation resulting from the requirement that the Poisson bracket between the Hamiltonian H and the second invariant I_2 vanishes. However, there is no general method to solve the partial differential equation and therefore one must make a simplifying *Ansatz* for I_2 and/or H. So far it has always been assumed that the second invariant is *polynomial* in momenta. Some results for two-dimensional systems have been given recently.[1-8] (The maximum order for which an explicit invariant has been found is p^6 for some Toda[6] and Holt[5,8] potentials.)

In this Letter I present three two-dimensional Hamiltonian models, whose second invariant is either a *rational*, an *elementary transcendental*, or a *higher transcendental* function in momenta. These models were found with the *Ansatz* that the second invariant depends only on two independent variables. We will also construct a third invariant I_3, canonically conjugate to I_2 (i.e., $[I_2,I_3]_P=1$). For the first two examples we will be able to write down the motion as an explicit function of time. Finally we briefly look at how these models behave with respect to the Painlevé integrability test.

Model A.—The Hamiltonian is given by

$$H_A^A=\tfrac{1}{2}p_x^2+\tfrac{1}{2}(p_y-x/y)^2-\tfrac{1}{2}x^2/y^2. \qquad (1)$$

This describes the motion of a charged particle of unit mass in a fictitious electromagnetic field, whose vector potential is $\vec{A}=(0,x/y)$ and scalar potential $\phi=\frac{1}{2}x^2/y^2$. There is a singularity at $y=0$. The equations of motion are

$$\ddot{x}=y^{-1}\dot{y}+y^{-2}x,\quad \ddot{y}=-y^{-1}\dot{x}-y^{-3}x^2. \qquad (2)$$

The Hamiltonian (1) has the second invariant

$$I_2^A=(xp_y-yp_x+y)/p_y, \qquad (3)$$

which is rational in momenta. A third invariant can also be constructed for this model; I have found

$$I_3^A=p_x+\ln(p_y/y). \qquad (4)$$

Of course many other pairs of invariants can be constructed from these. For example the logarithmic singularity in (4) can be eliminated by considering $\exp I_3^A$. One could also try to find some function of H, I_2^A, and I_3^A which would provide us a nontrivial second invariant *polynomial* in momenta. (In Ref. 7 it was conjectured that polynomial invariants can be constructed from rational and exponen-

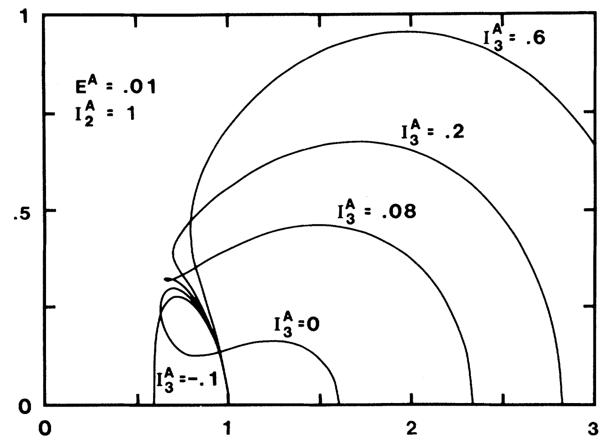

FIG. 1. Some trajectories for model A. I_2^A and E^A are kept constant and I_3^A varied. All trajectories fall eventually to the $y=0$ singularity.

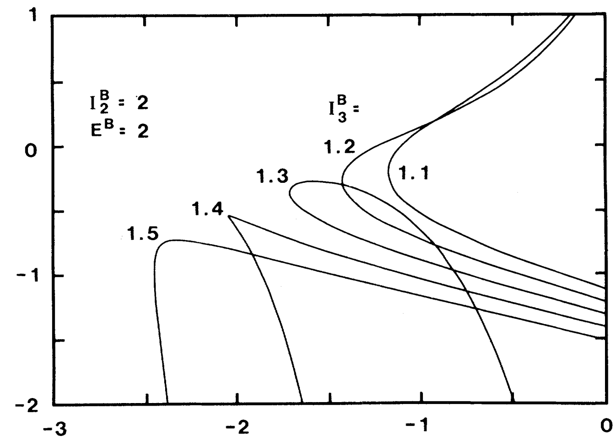

FIG. 2. Some trajectories for model B. I_2^B and E^B are kept constant and I_3^B varied. See text for behavior of $t \to \pm\infty$.

tial invariants.) However, this is not possible. First observe that H^A, I_2^A, and I_3^A are relatively prime as polynomials in p_x, and therefore, if the new invariant is to be a polynomial in p_x it must be polynomial in H^A, I_2^A, and I_3^A. This eliminates I_3^A dependence, as it is transcendental, and it is clear that no nontrivial polynomial of H^A, I_2^A is a polynomial in both p_x and p_y. Since this is a two-dimensional system we cannot have three mutually commuting invariants. In fact we have $[I_2^A, I_3^A]_P = 1$, i.e., I_2^A and I_3^A are a canonically conjugate pair.

Using the above results we can solve for the motion of the particle:

$$x(t) = I_2^A + (t-\tau)\ln(t-\tau) + (t-\tau)I_3^A,$$
$$y(t) = \pm\{2I_2^A(t-\tau) + (t-\tau)^2[2E^A - 1 - (\ln(t-\tau) + I_3^A - 1)^2]\}^{1/2}, \tag{5}$$

where E_A, I_2^A, and I_3^A now stand for the constant values of these invariants. Some trajectories are given in Fig. 1. For $I_2^A > 0$ the particle emerges from the $y=0$ singularity parallel to the y axis. The trajectory has a cusp singularity whenever $E^A = [3 - I_2^A \exp(1+I_3)]/2$.

Model B.—This is defined by the Hamiltonian

$$H^B = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + 2yp_xp_y - x. \tag{6}$$

The second and third invariants for H^B are transcendental:

$$I_2^B = p_y \exp(p_x^2), \quad I_3^B = -y\exp(-p_x^2) + \tfrac{1}{4}(2\pi)^{1/2}p_y \exp(p_x^2)\operatorname{erf}(\sqrt{2}p_x). \tag{7}$$

Again $[I_2^B, I_3^B] = 1$ and it is clear that there is no invariant algebraic in momenta, other than H^B itself. The equations of motion are

$$\ddot{x} = 2(2\dot{x}y - \dot{y})^2(4y^2-1)^{-2} + 1, \quad \ddot{y} = 4y(\dot{x} - 2\dot{y}y)^2(4y^2-1)^{-2} + 2y, \tag{8}$$

and the explicit solution with the invariants as constant parameters is

$$x(t) = -E^B - 2I_2^BI_3^B(t-\tau) + \tfrac{1}{2}(t-\tau)^2 + \tfrac{1}{2}(I_2^B)^2\{\exp[-2(t-\tau)^2] + (t-\tau)(2\pi)^{1/2}\operatorname{erf}[\sqrt{2}(t-\tau)]\},$$
$$y(t) = \exp[(t-\tau)^2]\{-I_3^B + I_2^B \times \tfrac{1}{4}(2\pi)^{1/2}\operatorname{erf}[\sqrt{2}(t-\tau)]\}. \tag{9}$$

Some orbits are given in Fig. 2. There is again the possibility of a cusp singularity for certain parameter values. The behavior at $t \to \pm\infty$ is also interesting. While $x \sim t^2/2$ at both limits we find that $y \sim \exp(t^2)[-I_3^B \pm \frac{1}{4}(2\pi)^{1/2}I_2^B]$ as $t \to \pm\infty$. Thus the parameter values related by $I_3^B = \pm\frac{1}{4}(2\pi)^{1/2}I_2^B$ are bifurcation points in the respective limits. Both Figs. 1 and 2 also suggest an enveloping curve to the displayed family of trajectories.

Model C.—The Hamiltonian is now

$$H^{C}=\tfrac{1}{2}p_x^2+\tfrac{1}{2}p_y^2+x/y, \tag{10}$$

which is of the usual "kinetic energy + potential" type. For this Hamiltonian the second (and third) invariants can only be expressed in terms of the parabolic cylinder functions.[9] Let $W_+(a,x)$ and $W_-(a,x)=W_+(a,-x)$ be the two independent standard solutions of the equation $y''(x)+(\frac{1}{4}x^2-a)y(x)=0$. The Wronskian of W_+ and W_- is equal to unity. In terms of W_+ and W_- the second and third invariants are given by

$$I_2^C=\tfrac{1}{2}y\,[p_y W_+(\tfrac{1}{2}E,p_x)+2W'_+(\tfrac{1}{2}E,p_x)]^2,\quad I_3^C=\frac{p_y W_-(\frac{1}{2}E,p_x)+2W'_-(\frac{1}{2}E,p_x)}{p_y W_+(\frac{1}{2}E,p_x)+2W'_+(\frac{1}{2}E,p_x)}. \tag{11}$$

Here the prime denotes differentiation with respect to p_x and E stands for the Hamiltonain H^C. Again $[I_2^C,I_3^C]=1$. Since the parabolic cylinder functions are entire analytic functions for all values of the parameter we see that I_2^C is a globally defined, single-valued function, as required.

The Newton's equations are

$$\ddot{x}=-1/y,\quad \ddot{y}=x/y^2, \tag{12}$$

but now we have not been able to construct x and y as explicit functions of time; instead we present here the trajectory with p_x as a parameter:

$$\begin{aligned} x(p_x)&=-I_2^C(p_x/4+1)[W'_-(\tfrac{1}{2}E,p_x)-I_3^C W'_+(\tfrac{1}{2}E,p_x)]^2+\tfrac{1}{2}E[W_-(\tfrac{1}{2}E,p_x)-I_3^C W_+(\tfrac{1}{2}E,p_x)]^2,\\ y(p_x)&=\tfrac{1}{2}I_2^C[W'_-(\tfrac{1}{2}E,p_x)-I_3^C W'_+(\tfrac{1}{2}E,p_x)]^2. \end{aligned} \tag{13}$$

The Painlevé analysis[10] has been used successfully to predict integrability.[2,4-6] For most of the Hamiltonian models having the "weak Painlevé" property one has been able to find a second invariant, polynomial in momenta.[4-6] Now that we have integrable models with rational or transcendental invariants it would be interesting to see what could have been predicted from a Painlevé analysis. It turns out that none of the models presented here have solutions with a divergent leading behavior, which is usually assumed in a Painlevé analysis.

For model A we start with the *Ansatz* $x=at^{\mu}+\dots$, $y=bt^{\nu}+\dots$ for the leading behavior and substitute into (2). We find first

$$a\mu(\mu-1)t^{\mu-2}=\nu t^{-1}+(a/b^2)t^{\mu-2\nu},\quad b\nu(\nu-1)t^{\nu-2}=-(a/b)\mu t^{\mu-\nu-1}-a^2b^{-3}t^{2\mu-3\nu}. \tag{14}$$

Here the leading terms can be made to cancel if we choose (1) $\mu=\nu=1$, $a=-b^2$, b arbitrary or (2) $\mu=0$, $\nu=\frac{1}{2}$, $a=-1/2b^2$, b arbitrary. Next we would have to study the resonances, but we stop here, because *neither* of these gives the beginning of the correct expansion. With hindsight from (5) we note that the nonleading $t\ln t$ term can also produce a t^{-1} or $t^{-3/2}$ term in the left-hand side of (14). The correct expansion starts with $\mu=0$, $\nu=\frac{1}{2}$, $a=+1/2b^2$, b arbitrary. Presumably we would have found this problem had we studied the resonances. The point is that when the leading behavior is not divergent nonleading terms can contribute through the derivative terms.

The equations (8) for model B are somewhat more difficult to study, but they do not seem to follow any singular behavior either.

Equations (12) for model C are best studied if x is first solved in terms of y and y''. The equation for y is then

$$y^3y^{\text{iv}}+4y'''y'y^2+2(y'')^2y^2+2y''y'^2y+1=0.$$

The leading behavior for y can now be $y=at^{\mu}$ with $\mu=0$, $\frac{2}{3}$, or 1, or $y=(2i)^{1/2}t(\ln t)^{1/2}$. The first is a regular expansion having resonances at 0, 1, 2, and 3:

$$y(t)=A+Bt+Ct^2+Dt^3-(A^2BD+\tfrac{1}{3}A^2C^2+\tfrac{1}{6}AB^2C+\tfrac{1}{24})A^{-3}t^4+\dots.$$

For the second case we write $y(t)=(t-\tau)^{2/3}[A+B(t-\tau)^{\rho}]$ and then there are resonances at $\rho=-1$, 0, $\frac{2}{3}$, and 1, which is typical of what one obtains in a Painlevé analysis. The expansion starts now according to

$$y(t)=(t-\tau)^{2/3}\{B_2+B_4(t-\tau)^{2/3}+B_5(t-\tau)-[\tfrac{3}{7}B_4^2B_2^{-1}-\tfrac{81}{56}B_2^{-3}]t^{4/3}+\dots].$$

The third case has only double poles; we have not studied further the fourth one.

None of these models has an expansion with a divergent leading term and, as we have seen, branch cuts and logarithims are found. Thus these models do not pass the conventional Painlevé test, although models B and C come very close. Nevertheless all models have expansions where the necessary four parameters can enter. If one wants to make a new conjecture on the basis of these results one could say that the critical property is the existence of enough resonances (four in this case).

The Painlevé test has often been used successfully to find candidates for integrable models, but for the final proof of integrability one usually has to construct the second invariant explicitly. For obvious reasons the necessary search for invariants has usually been made among polynomials of p_x and p_y. The above results suggest, however, that an extension to transcendental invariants would be productive. The systematic methods for doing this are still to be formed.

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