Fractal Dimension of Dielectric Breakdown

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It is shown that the simplest nontrivial stochastic model for dielectric breakdown naturally leads to fractal structures for the discharge pattern. Planar discharges are studied in detail and the results are compared with properly designed experiments.

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Dielectric breakdown of gaseous, liquid, and solid insulators frequently occurs by means of narrow discharge channels that exhibit a strong tendency to branching into complicated stochastic patterns.[1] Examples are lightnings,[2] surface discharges (Lichtenberg figures),[3] and treeing in polymers.[4] The global structure of branched discharges often shows a close structural similarity within a large class of discharge types but at the moment even a qualitative classification of these structures is missing.

In this Letter we present some evidence for fractal properties of branched discharges by analyzing adequately designed experiments and by the study of a new theoretical model. In particular we show that the simplest nontrivial stochastic model of dielectric breakdown naturally leads to fractal structures.

In order to facilitate the analysis of an experimental discharge pattern the optimal situation is that of a two-dimensional radial discharge like the one shown in Fig. 1. This example refers to a leader surface discharge (Lichtenberg figure) in compressed SF_6 gas whose properties have been studied in detail previously by Niemeyer and Pinnekamp.[5] The parameters were controlled in such a way that the experiment produces, to a good approximation, an equipotential channel system growing in a plane with a radial electrode from a central point.

If Fig. 1 corresponds to a fractal structure[6] the relation between the total length of all branches (or total number of points in the sense of Ref. 6) inside a circle of radius r and the radius r itself should be a power law with noninteger exponent D:

$$N(r) \sim r^D. \qquad (1)$$

The thickness of the branches is considered as zero dimensional; it does not grow with the size of the object. Note that the apparent thickness in the photo of Fig. 1 is just an optic effect due to the number of carriers that have passed through a given branch. The number of branches $n(r)$ at a given distance r from the center is then given by

$$n(r) \sim dN(r)/dr \sim r^{(D-1)}. \qquad (2)$$

Therefore a careful counting[7] of the number of branches for different values of r provides information about the exponent D. The analysis of the photos suggests a power-law behavior with $D \sim 1.7$, but the finite size and resolution do not allow an accurate determination of this exponent.

In order to gain insight into which features of the phenomenon of dielectric breakdown are the relevant ones with respect to its fractal properties we introduce an appropriate stochastic model and study it by computer simulations. We consider a two-dimensional square lattice (Fig. 2) in which a central point represents one of the electrodes while

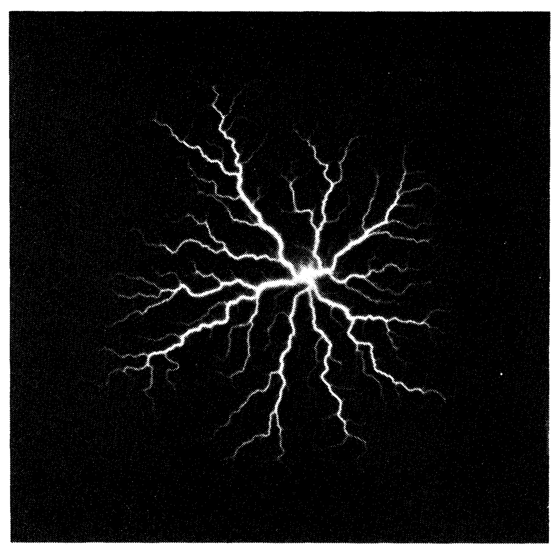

FIG. 1. Time-integrated photograph of a surface leader discharge (Lichtenberg figure) on a 2-mm glass plate in 0.3-MPa SF_6. Applied voltage pulse: 30 kV$\times$1 μs (Ref. 5). This experiment corresponds to an equipotential channel system growing in a plane with radial electrode.

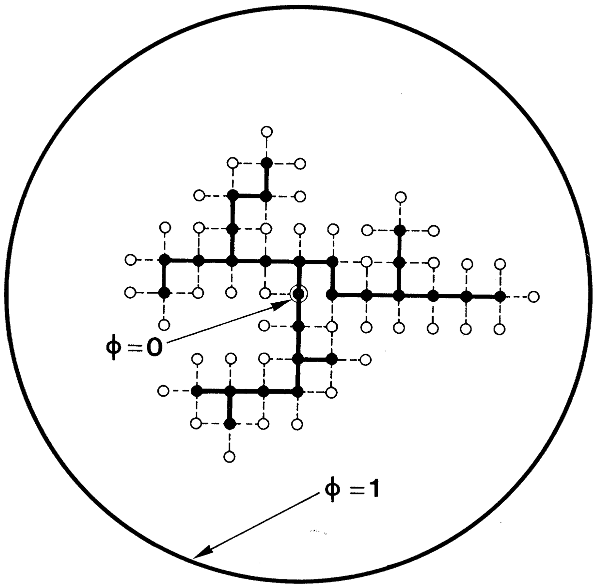

FIG. 2. Illustration of the stochastic model we introduce to simulate dielectric breakdown on a lattice. The central point represents one of the electrodes while the other electrode is modeled as a circle at large enough distance. The discharge pattern is indicated by the black dots connected with thick lines and it is considered equipotential ($\phi = 0$). The dashed bonds indicate all the possible growth processes. The probability for each of these processes is proportional to the local electric field (see text).

the other electrode is modeled as a circle at large enough distance. The rules we assume for the growth of the discharge pattern are the following:

(a) The pattern grows stepwise. Figure 2 gives an example of a configuration after a few steps: The discharge pattern is indicated by the black dots connected with thick lines. The electric potential ϕ is defined for all points of the lattice by the discrete Laplace equation with the boundary conditions $\phi = 0$ for each point of the discharge pattern and $\phi = 1$ outside the external circle.

(b) At each step one bond is added to the pattern, linking a point of the pattern with a new point. The possible candidates are indicated in Fig. 2 by the dashed bonds that link a black point to a white one.

(c) To each of these dashed bonds a probability p is associated that is a function of the potential difference (local electric field[8]) between the black (i,k; $\phi_{ik} = 0$) and white (i',k') dots connnected by this bond. The indices i,k and i',k' represent the discrete lattice coordinates. We write

$$p(i,k \rightarrow i',k') = \frac{(\phi_{i',k'})^{\eta}}{\sum(\phi_{i',k'})^{\eta}}, \tag{3}$$

where a power-law dependence with exponent η is assumed to describe the relation between local field and probability. The sum in the denominator refers to all the possible growth processes (dashed lines in Fig. 2). Given this probability distribution a new bond (and point) is chosen randomly and added to the discharge pattern. With this new discharge pattern one starts again. These rules properly define also the starting of the process from the central point. Further it follows that no crossing is possible and that the pattern is simply connected.

The essence of this stochastic model is therefore that the growth probability depends on the local field (potential) determined by the equipotential discharge pattern. The most laborious part of this program is the solution, at each time step, of the Laplace equation

$$\nabla^2 \phi = 0. \tag{4}$$

The discrete form of Eq. (4) on the two-dimensional lattice[9] can be written as

$$\phi_{i,k} = \tfrac{1}{4}(\phi_{i+1,k} + \phi_{i-1,k} + \phi_{i,k+1} + \phi_{i,k-1}). \tag{5}$$

Given the appropriate boundary conditions the potential is obtained by iterating Eq. (5). Typically good convergence is obtained with a number of iterations between 5 and 50. This method correctly reproduces the global influence of a given discharge pattern on the growth probability for each bond. So for example, the tip of a line like the one on the right side of Fig. 2 will have a large growth probability while the points inside a cage (e.g., left side of Fig. 2) will have much smaller probability. These are the well known "tip effect" and "Faraday screening" that result from the solution of the field equation.

Recently Sawada *et al.*[10] have tried to simulate dielectric breakdown by simply assigning *a priori* a larger probability for the growth of tips with respect to side branching. This simplification completely neglects the nonlocal nature of the electric field. For example a tip inside a Faraday cage has in their approach the same growth probability as a free tip. The results obtained with this approximation are qualitatively different from experimental discharges and their global dimensionality remains the Euclidean one ($D = 2$). Badaloni and Gallimberti[11] instead do not approach the problem of the dimensionality of the structure but provide an interesting connection between a microscopic picture of the avalanche formation and the bifurcation probability and geometry at a tip.

We start the discussion of the computer simulations with the case $\eta = 1$ (growth probability proportional to the local field) that is the most realistic case for the present experiment.[5] We obtain highly branched structures like the one shown in Fig. 3. In order to investigate the dimensionality of these discharge structures we have generated several of them and we have plotted the logarithm of the total number of points within a certain radius as a function of the logarithm of this radius. The average over five large samples of about 5000 points each gives rise to a Hausdorff dimension $D = 1.75 \pm 0.02$. The error bar is due to the statistical fluctuations of the slope but the possibility of a larger systematic error due to the finite size of the systems considered cannot be excluded. We can see that the computed value is in good agreement with the value $D \simeq 1.7$ suggested by the analysis of the experimental discharges. This comparison anyhow should be considered with caution because in the computer simulations we have two-dimensional field equations while in the experiment only the growth is two-dimensional while the field lines extend three dimensionally. This is a point that we intend to consider in more detail in future work.

We study further the effect of values of η different from 1. This is of interest because in other systems than gases (solids, liquids, and polymers) the microscopic relation between growth probability and local field may be more appropriately described by a nonlinear function.[1,4]

The case $\eta = 0$ corresponds to growth probabilities independent on the local field. This is a sort of Eden (cancer) model[12] with the difference that some of the white points in our case (Fig. 2) have a higher probability because they can be reached from more than one black point. The growth is in this case homogeneous with $D = 2$. We have further considered the cases $\eta = 0.5$ and $\eta = 2$. The corresponding Hausdorff dimensions are reported in Table I and they differ from the case $\eta = 1$. This shows that the Hausdorff dimension depends on the parameter η and there is no universality in this respect. For large η the structure tends to be more "linear," but a careful study of the limit $\eta \to \infty$ is not possible with the present method. We suggest, however, that, in contrast with the simple behavior for $\eta \to 0$, the limit $\eta \to \infty$ is likely to be nonanalytical in the sense that the exchange of limits $\eta \to \infty$ and $N \to \infty$ may give rise to different behaviors.[14]

Another interesting variation we have considered is the effect of a Debye screening (due to eventual mobile carriers in the system) or correspondingly a finite mass in a field theory. This is realized by replacing ∇^2 in Eq. (4) by $(\nabla^2 - k^2)$. Such a change introduces a length scale $(l \sim k^{-1})$ into the system that then loses its self-similarity. The results show nonhomogeneous patterns for length scales lower tha l but homogeneous behavior $(D = 2)$ on larger scales.

With respect to other growth models the general shape of our patterns for $\eta = 1$ (Fig. 3) and the corresponding value of D closely recall the results of the diffusion-limited aggregation models.[12,15-17] This similarity actually arises from the close relations between random walk and potential theory, a point to which we intend to dedicate a more extended discussion elsewhere.

In summary we have introduced a new stochastic model to describe the discharge pattern of dielectric breakdown. The basic assumption is that the growth probability depends on the local electric field. We show that this model naturally leads to fractal structures that we study in detail in two

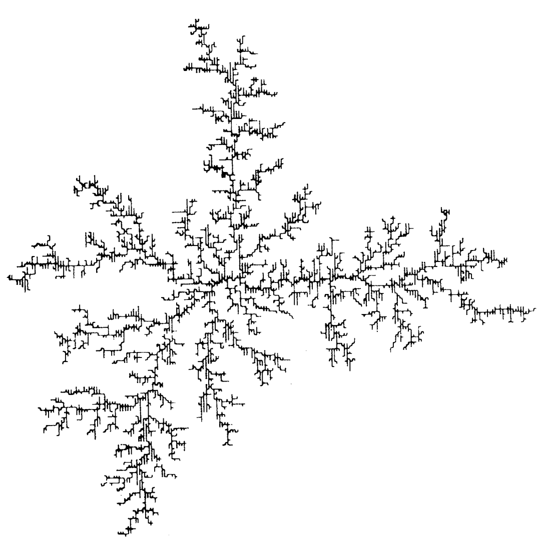

FIG. 3. Example of computer-generated discharge pattern of about 5000 steps. The Hausdorff dimension for these structures turns out to be $D = 1.75 \pm 0.02$.

TABLE I. Dependence of the Hausdorff dimension D on the exponent η used in the relation between probability and local field [Eq. (3)].

η	D
0	2
0.5	1.89 ± 0.01
1	1.75 ± 0.02
2	$\sim 1.6^{a}$

[a]Ref. 13.

dimensions. The results are consistent with the analysis of properly designed experiments of planar discharge.

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[7]We have counted all the branches visible in the original negative. Some tiny ones could disappear in the printed reproduction of the photo.

[8]The probabilistic approach reproduces the stochastic nature of the ionization avalanche formation that starts the discharge progression.

[9]Actually in the experiment only the discharge is planar while the field extends three dimensionally.

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[12]For an overview on various models of aggregation see, H. E. Stanley, F. Family, and H. Gould, to be published.

[13]The case $\eta = 2$ gives rise to patterns with scarce ramifications for which one should probably consider larger sizes in order to obtain a reliable value for D.

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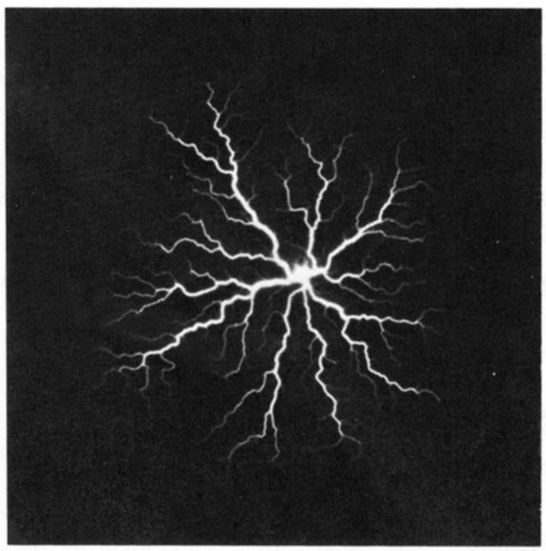

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