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Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

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It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

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The statistical study of spectra of quantum systems is almost as old as quantum mechanics itself. One distinguishes two types of properties: *global* ones and *local* ones. An example of the former is provided by the density of levels as a function of excitation energy. In this Letter we shall discuss local properties, or more precisely, fluctuations (departures of the energy-level distribution from uniformity). We shall deal with time-independent systems and energies of stationary states.

There exists a well established theory to describe fluctuation properties of quantal spectra, namely the random matrix theory (RMT) initiated by Wigner, developed mainly by Dyson and Mehta,[1] and later extended by several authors.[2] Recently,[3] the predictions of RMT [specifically, the predictions of the Gaussian orthogonal ensemble (GOE)] have been compared in great detail with the whole body of available nuclear data coming mainly from *compound-nucleus resonances*. No discrepancy between theory and experiment has been detected. In particular the data have been shown to exhibit two of the salient phenomena predicted by the theory—the *level repulsion* (tendency of the levels to avoid clustering) and especially the *spectral rigidity* (very small fluctuation around its average of the number of levels found in an interval of given length), which is a property due to correlations between level spacings. We should also mention that recently a comparison between data of *atomic levels* and GOE has been performed.[4] Although the significance of the comparison is lower than in the nuclear case (due to the method used in Ref. 4 as well as to the relatively small number of available data), a good agreement between GOE predictions and experiment was found.

Once the ability of the theory (GOE) to predict the fluctuation properties exhibited by data is established, one could think that no major question in this field is still open. It is one purpose of this Letter to show that this is not the case. Indeed, as it will be discussed in what follows, the origin of the success of the theory as well as its domain of validity remain to be clarified.

In connection with the study of regular and irregular motion some very interesting results have been obtained in recent years. Strong arguments have been given which indicate that for integrable systems with more than one degree of freedom,[5] the nearest-neighbor spacing distribution $p(x)$ of the quantum energy levels should be Poisson-like $[p(x)=\exp(-x)]$ and the spacings should not be correlated.[6,7] In contrast, evidence of level repulsion has been put forward by studying numerically systems having two degrees of freedom and known to be chaotic[8] in the classical case (stadium,[9] Sinai's billiard[10]). Although this feature is clearly appealing, it is up to now only qualitative [one would need a spectrum with very many levels to get a precise evaluation of the behavior of $p(x)$ at the origin]. On the other hand, the information carried by $p(x\sim 0)$ is rather limited. In particular, it does not give any indication about the *correlations* between spacings which are responsible for the degree of regularity of the spectrum. The purpose of this Letter is to use some of the systematic tools developed in RMT to make a detailed comparison of the level fluctuations of the quantum Sinai's billiard (SB) with GOE predictions. The choice of a two-dimensional billiard is convenient for our aim for several reasons: (i) Billiards have the lowest possible number of degrees of freedom allowing for chaotic motion; (ii) for billiards, it is possible to make a precise separation between global and local properties [cf. the Weyl formula, Eq. (1)]; (iii) billiards have a discrete spectrum with an infinite number of eigenvalues and by computing a large number of them one can reach a high statistical significance of the results. Finally, SB is known to be strongly chaotic (K system) and there exists an efficient method to compute its eigenvalues.

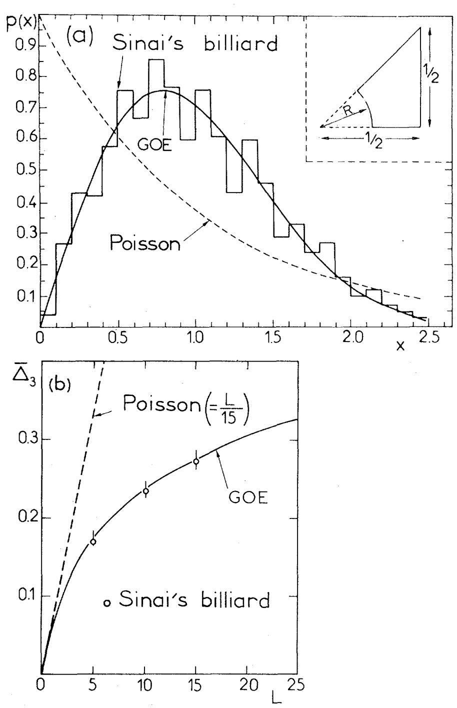

FIG. 1. Results of energy-level fluctuations for desymmetrized Sinai's billiards as specified in the upper right-hand corner of (a). 740 levels have been included in the analysis corresponding to the 51st to 268th level for $R=0.1$, 21st to 241st level for $R=0.2$, 16th to 194th level for $R=0.3$, 11th to 132nd level for $R=0.4$. (a) Results for the nearest-neighbor spacing distribution. (b) Results for the average value of the $\Delta_3(L)$ statistic of Dyson and Mehta for $L=5$, 10, and 15. Curves corresponding to the Poisson case (stretch of uncorrelated levels) and to the random-matrix–theory predictions (GOE) are drawn for comparison. The error bars in (b) (one standard deviation) correspond to finite-sampling effects as predicted by GOE.

We proceed as follows. We determine the eigenvalues $E_n = k_n^2/2m$ of the Schrödinger equation $(\Delta + k_n^2)\psi_n = 0$ for the "desymmetrized" SB [see upper right-hand corner of Fig. 1(a)] with Dirichlet boundary conditions by using the Korringa-Kohn-Rostoker method as described in Ref. 10. We compute several sets of eigenvalues $\{E_i(R)\}$ for different values of the parameter R (see caption of Fig. 1). By using the Weyl-type formula,[11] which gives the average number of levels up to energy E,

$$\bar{N}(E) = (\tfrac{1}{4}\pi)(SE - L\sqrt{E} + K), \qquad (1)$$

where S and L are, respectively, the surface and the perimeter of the billiard and K is a constant of the order of unity, we can map the spectrum $\{E_i(R)\}$ onto the spectrum $\{\epsilon_i(R)\}$ through $\epsilon_i(\boldsymbol{R}) = \bar{\boldsymbol{N}}(E_i(\boldsymbol{R}))$. Each spectrum $\{\epsilon_i(R)\}$ has on the average a constant mean spacing $D(R)$ which will be taken as the energy unit. The cumulative density $n(\epsilon)$ of levels ϵ_i will therefore have a staircase shape fluctuating around a straight line of slope equal to unity. In order to investigate the fluctuations we study the nearest-neighbor spacing distribution $p(x)$ and the Dyson-Mehta statistic Δ_3. Δ_3 is defined, for a fixed interval $[x, x+L]$, as the least-squares deviation of the staircase function $n(\epsilon)$ from the best straight line fitting it:

$$\Delta_3(L,x) = (1/L)\mathrm{Min}_{A,B}\int_x^{x+L}[n(\epsilon) - A\epsilon - B]^2\, d\epsilon. \qquad (2)$$

It provides a measure of the degree of rigidity of the spectrum: For a given L, the smaller Δ_3 is, the stronger is the rigidity, signifying the long-range correlations between levels. We proceed as described in Ref. 3: Given a stretch of levels on the ϵ axis, we compute $\Delta_3(L)$, for instance, for the intervals $[a, a+L]$, $[a+L/2, a+3L/2]$, $[a+L, a+2L]$, $[a+3L/2, a+5L/2]$,... until the stretch $[a,b]$ has been covered. If the spectrum fluctuations are translationally invariant on the ϵ axis, then the average value $\bar{\Delta}_3$ of Δ_3 will be independent of the chosen interval $[a,b]$ [equivalently, the average of Δ_3 computed with Eq. (2) will not depend on the initial value x of the interval]. We have checked numerically that this is the case for the SB spectra, provided that the first lowest levels of each spectrum $\{\epsilon_i(R)\}$ are omitted (see caption of Fig. 1). This procedure seems appropriate if one observes the spectra corresponding to different values of R: Indeed, the smaller the value of R the larger the number of levels, starting from the ground state, that can be attained by perturbation theory from the spectrum $\{\epsilon_i(R=0)\}$ (triangular billiard). We emphasize that the fluctuation properties we are looking for will appear to be essentially different from the ones corresponding to the triangular billiard[12] and cannot be attained from it by perturbation theory. To increase the statistical significance of the results, four spectra $\{\epsilon_i(R)\}$ corresponding to different values of R will be analyzed as corresponding to a single stretch of 740 levels (see caption of Fig. 1). Care has been taken that one is dealing with "independent information": The different values of R should not be chosen too close to one another. Otherwise, two different spectra corresponding to R and $R+\delta R$ would be almost deducible one from the other and one would just be analyzing redundant information.

Let us now discuss the results. In Fig. 1(a) is shown the nearest-neighbor spacing distribution $p(x)$ which is compared to the GOE and Poisson predictions. As can be seen, the SB results follow very closely GOE not only for small spacings (level repulsion) but over the whole range of spacings. The variance of $p(x)$ for SB is 0.273 which is close to the GOE value 0.286 ± 0.015 (the error bar takes into account the finite sampling effects) and far from the Poisson value 1.0. We next consider quantities related with the spacing correlations. In Fig. 1(b) are shown the average values of $\Delta_3(L)$ for $L=5$, 10, and 15 for SB; they are close to the corresponding GOE values. We have also computed the correlation factor between two adjacent spacings. For SB we obtain -0.30, to be compared to -0.27 ± 0.04 (GOE) and 0.0 (Poisson). We can summarize the numerical results as follows: All fluctuation properties of SB investigated so far are fully consistent with GOE predictions.

Is this a surprising result? With a few inconclusive exceptions (see a discussion on small metallic particles, for instance in Ref. 2), the basic hypotheses leading to RMT have always been put forward by invoking the complexity of the system. In other words, it has been taken as essential that one is dealing with a many-particle system (system with many degrees of freedom). Our results indicate that this is by no means a necessary condition. Indeed, the quantum chaotic system with two degrees of freedom studied here (a one-particle system in two dimensions) shows also GOE fluctuations. The present work should have further developments [for instance, when time-reversal invariance does not hold, the adequate model in RMT is the Gaussian unitary ensemble (GUE) and one should look for "simple" chaotic systems having GUE fluctuations]. It is an attempt to put in close contact two areas—random matrix physics and the study of chaotic motion—that have remained disconnected so far. It indicates that the methods developed in RMT to study fluctuations provide the adequate tools to characterize chaotic spectra and that, conversely, the generality of GOE fluctuations is to be found in properties of chaotic systems. In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal–invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE (alternative stronger conjectures that cannot be excluded would apply to less chaotic systems, provided that they are ergodic). If the conjecture happens to be true, it will then have been established the *universality of the laws of level fluctuations* in quantal spectra already found in nuclei and to a lesser extent in atoms. Then, they should also be found in other quantal systems, such as molecules, hadrons, etc.

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[8]We use here the term chaotic in a qualitative way. In the classical case it designates a large class of systems whose trajectories are unstable with respect to the initial conditions. In fact, the specific systems we call chaotic in the Letter are K systems. Via the correspondence principle we cover the quantum case, for which the asymptotic behavior of the spectrum at high energies is of most relevance.

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