Additional Radiative-Recoil Corrections to Muonium and Positronium Hyperfine Splitting

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A compact expression for one-loop radiative corrections to lepton lines in positronium and muonium hyperfine splitting is presented. It is valid for hyperfine-splitting contributions of order $\alpha^2 E_F$, including the effects of recoil to all orders. For muonium, known nonrecoil parts of this contribution are easily evaluated analytically, as is a previously calculated radiative-recoil contribution of order $\alpha^2 E_F (m_e/m_\mu)\ln(m_\mu/m_e)$. Nonlogarithmic radiative-recoil corrections obtained by numerical evaluation of this expression yield new contributions of 2.64 ± 0.07 kHz for muonium and -11.12 ± 0.02 MHz positronium.

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The largest correction to the lowest-order hyperfine splitting (hfs) arises from the Schwinger correction to the magnetic moment of the electron. It results from the emission and reabsorption of a virtual photon by the electron. The bulk of this effect can be taken into account by simply multiplying the leading contribution E_F by $\alpha/2\pi$. Other corrections arise because the electron is in a bound state. Thus they are sensitive to the internal structure of the anomalous moment as well as other binding effects; they produce additional powers of α.[1,2] Furthermore, the electron is bound to a particle of finite mass. This affects the hfs in two ways. One is through the reduced mass which simply modifies the magnitude of the wave function at small distances. The other is that it produces dynamical recoil corrections, which are smaller than E_F by the mass ratio m_e/m_μ, as well as one or more powers of α. This paper is concerned with the combination of radiative and recoil corrections, which we call radiative-recoil corrections.[3] A previous paper gave the results for such terms containing powers of $\ln(m_\mu/m_e)$.[4] There it was suggested that the additive constants would be relatively small. We find here that they are larger than previously expected.

The Schwinger correction can be expressed in terms of the reduced mass and it involves no dynamical recoil effects. The latter effects appear when the internal structure of the anomalous moment and the other contributions associated with Fig. 1 are taken into account.[5] Since the vacuum polarization contribution has been adequately discussed previously,[4] this paper deals only with the contributions illustrated in Fig. 1. Their leading nonrecoil correction of relative order α^2 is $\alpha^2 E_F(\ln 2 - \frac{13}{4})$. The subject of this paper is the setting up of an expression valid to this order in α and to all orders in the mass ratio m_e/m_μ. This approach is essential for application to positronium where no expansion in a small mass ratio exists, and it is very useful for the computation of the additive constants associated with the previously determined $(m_e/m_\mu)\ln(m_\mu/m_e)$ terms.

We now present the most salient features of the calculation, leaving a more detailed description to a later publication. Care must be taken that contributions from graphs in which the virtual photon spans additional Coulomb interactions are of a higher order than those considered here. It is possible that such graphs contribute in the order of interest because the electron is close to its mass shell and integrations can yield inverse powers of α. To control this infrared sensitivity, we use the Fried-Yennie gauge[6] while keeping the electron off mass shell; then spurious terms of lower order explicitly cancel, insuring that these

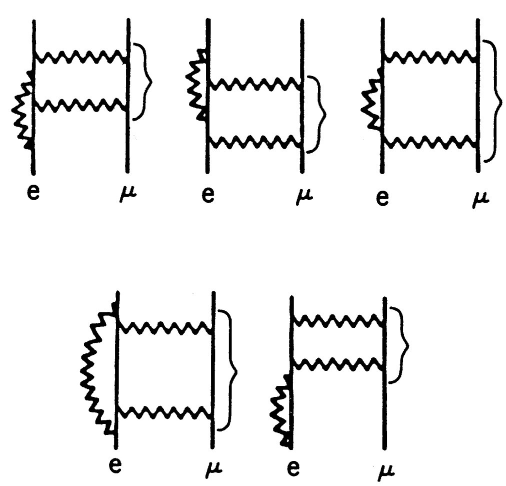

FIG. 1. Graphs contributing to the radiative correction on the electron line. A symmetric set occurs for the muon line.

contributions are of higher order than $\alpha^2 E_F$.

There are several sources of simplification. A great many numerator terms may be neglected since we need keep only Dirac structures which contribute to the hyperfine splitting. Also we restrict our attention to terms of order $\alpha^2 E_F$, which allows the neglect of wave-function momenta in comparison to loop momenta. The result is that the wave-function dependence decouples from the rest of the calculation and produces a factor which is the square of the nonrelativistic wave function at the origin. This factor depends on the reduced mass m_r. The internal integrations of the kernel produce a factor of $\alpha^2 E_F$ times some function of m_e and m_μ. The general features of the analysis of the loop integrations will be described briefly.

First it should be recalled[5] that the contributions under consideration contain pieces of the same order as the Schwinger contribution. Because of the approximation of neglecting the wave-function momenta inside the kernel, these pieces will not precisely reproduce the Schwinger contribution. We simply identify and discard them since that does not affect the calculation at the present order of interest. For muonium, they also include the nonrecoil terms of order $\alpha^2 E_F$. We identify and evaluate these as a check on our formalism. The calculation is relatively simple. Then we devise a numerical procedure which eliminates them since they are much larger than the terms we seek. This procedure is not relevant for positronium since the two effects are of the same order of magnitude.

The actual techniques will now be outlined. We first study the five contributions in momentum space and find various cancellations between them. This has the great advantage of eliminating spurious $\ln^2(m_\mu/m_e)$ terms. Next the integration over the loop containing the virtual photon is carried out at the expense of the introduction of integrations over Feynman parameters x and y. Contributions from all the graphs are combined and rearranged by making extensive and judicious use of partial integrations with respect to x and y. The result may be expressed in various forms. One of the forms which we have found useful is presented here to give the flavor of the calculation:

$$E = 2i\alpha^2 E_F \frac{m_e m_\mu}{\pi^3} \int_0^\infty \vec{p}^2 d|\vec{p}| \int_{-\infty}^\infty dp_0 \int_0^1 dx \int_0^1 dy \frac{1}{p^2} \frac{1}{(p^2)^2 - 4(p_0 m_\mu)^2} [(2\vec{p}^2 - 3p_0^2)T_1 + 3p_0 T_2], \tag{1}$$

where $p^2 = p_0^2 - \vec{p}^2 + i\epsilon$ and

$$T_1 = -\frac{2}{D_0} + \frac{F_1}{\Delta^2} + \frac{F_2}{\Delta^3}, \quad T_2 = \frac{2p_0}{D_0} + \frac{F_3}{\Delta^2} + \frac{F_4}{\Delta}, \quad D_0 = p^2 + 2m_e p_0, \quad \Delta = x m_e^2 - y(1-xy)p^2 - y(1-x)2m_e p_0.$$

The polynomials F are given by

$$\begin{aligned} F_1 &= 2m_e p_0 xy(-2 + 5y + 4x - 7xy) + yp^2(1 - 6x - 26y + 37xy + 6y^2 - 2x^2 - 12x^2y^2 + 2yx^2 - 16y\ln x), \\ F_2 &= 16m_e^2 \vec{p}^2 xy^2(1 - 3y - x + 3xy - 2y\ln x) + 4xy^2(p^2)^2[\ln x(2 - 8y + 8y^2) - (1-x)(1 - y - 2y^2)], \\ F_3 &= xym_e p^2(-7 + 4x + 8y - 2xy) - 2p_0 p^2 y^2(1-x)(1-y), \\ F_4 &= -yp^2(1-xy)/m_e. \end{aligned} \tag{2}$$

The terms with denominators D_0 arise from the static anomalous moment; aside from the Schwinger correction, they cancel each other to the order of interest. Here m_e and m_μ may be regarded as generic. They may be interchanged to give corrections in the muon line or set equal to give corrections for positronium.

The form given here is most convenient for the muon-line radiative corrections in muonium. The reason is that it was designed to make manifest the low-energy Compton scattering theorem. As $p \to 0$ it is dominated by the effects of the anomalous moment of the muon and other effects vanish. The nonanomalous moment terms are not sensitive to low-momentum details, and this renders them simpler to evaluate numerically. Other forms are more convenient for other purposes such as the evaluation of the electron-line radiative corrections. They will be presented in a more complete paper.

We have attempted the analytic evaluation of some of these integrals for certain terms where the radiative corrections are in the electron line and the muon propagator is rearranged in a convenient way which emphasizes the dominant terms in an expansion in powers of m_e/m_μ. We had expected these terms to be a simple numerical multiple of $(\alpha/\pi)^2(m_e/m_\mu)E_F$, but instead they were π^2 times such an expression. This perhaps accounts for the relatively large magnitude of our final result. While it might be feasible to calculate all terms of the order of interest analytically, the effort seems prodigious. Therefore we present here the results of a numerical evaluation of these integrals that allows extension of the calculation to positronium ($m_\mu = m_e$) where the analytic work is even less tractable. By rotating the p_0 contour ($p_0 \to ip_4$; crossing no poles or cuts), (1) becomes a real four-dimensional integral which we have evaluated by the adaptive Monte Carlo program VEGAS.[7] The results are presented in the following equations, where in muonium the $\ln(m_\mu/m_e)$ term previously calculated has been subtracted out to show the new contributions. The overall result for muonium is

$$E = 2.64 \pm 0.07 \text{ kHz}, \tag{3}$$

and for positronium

$$E = -11.12 \pm 0.02 \text{ MHz}. \tag{4}$$

While the result for positronium is only one of many terms of this order, the result for muonium should be the last contribution larger than a kilohertz. The discussion of the full experimental status is given in the following paper.[8]

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