Entropy from Extra Dimensions

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The phenomenon of "dynamical compactification" in a universe with $4+n_c$ dimensions is studied, and it is found that if the whole process is adiabatic, entropy is pumped into the effective four-dimensional universe. Some cosmological consequences of this fact are discussed.

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It is possible that space-time has more than four dimensions and that the extra ones have so far escaped detection.[1] The earliest suggestion in this respect is Kaluza and Klein's idea that the topology of an extra dimension is a circle S^1 of a very small radius, obtaining in this way a unified theory of gravitation and electromagnetism. The modern version[2] of this idea, called spontaneous compactification, proves that a manifold of the type $V_4 \times C_{n_c}$, where V_4 is the ordinary four-dimensional space-time and C_{n_c} is a compact space, is a static solution of Einstein's equations in $D \equiv 4+n_c$ dimensions and, as such, is as good a candidate for the vacuum of the quantum theory as D-dimensional Minkowski space.

In the event that the space-time has indeed extra dimensions, it is not sufficient to have proved that spontaneous compactification occurs; cosmological dynamics of the extra dimensions must also be studied,[3] explaining why the universe appears as four dimensional now, and what are the consequences of the disappearance, for all practical purposes, of the extra dimensions, while the known ones are well described by a Friedman-Robertson-Walker type of universe.

The main point to be made in this Letter is that it is possible to imagine in the framework of classical general relativity, a scenario in which the extra dimensions keep shrinking while the normal ones expand, and that if the whole process is adiabatic in the D-dimensional sense, a large amount of entropy can be produced. The production of entropy by this process can perhaps be viewed as somewhat similar to the sudden smashing of a D-dimensional lattice into a $D-1$ dimensional one, with the consequent increase in disorder. This process cannot go on forever, though. After the size of the extra dimensions becomes smaller than a critical size to be specified later, entropy production virtually stops and the expansion proceeds in the standard adiabatic way.

The entropy thus produced could eventually, depending upon the details of the solution, be as big as necessary to solve the horizon and flatness problems, providing hypothetical alternatives to the usual inflationary scenario.[4] Another possibility is that this entropy spreads out as uniform background radiation.

We will assume, for simplicity, that the metric has the Bianchi type-I form

$$ds^2 = dt^2 - R^2\delta_{ij}dx^i dx^j - A^2\delta_{ab}dy^a dy^b, \tag{1}$$

where $i,j=1,2,3$ (the ordinary space dimensions) and $a,b=4,\ldots,D$ (the compact dimensions).[5] The compact dimensions are "periodic" in the sense of being restricted to $-L \leq y_i \leq L$, where L is the constant fixing the scale of length in the extra dimensions. Here, $R(t)$ and $A(t)$ are functions of time only, to be determined from Einstein's equations $(m,n=0,\ldots,D)$

$$R_{mn} - \tfrac{1}{2}(R-2\lambda)g_{mn} = -8\pi G T_{mn}, \tag{2}$$

where the energy-momentum tensor is assumed to be of the perfect-fluid form

$$T_{mn} = (\rho+p)u_m u_n - p g_{mn}. \tag{3}$$

The integrability condition $\nabla_m T^{mn}=0$ implies a constraint among the functions $\rho(t)$, $p(t)$, $R(t)$, and $A(t)$:

$$\dot\rho + (\rho+p)(3\dot R/R + n_c\dot A/A) = 0 \tag{4}$$

where $\dot\rho \equiv d\rho/dt$, etc., and the pressure p is given in terms of ρ by an equation of state [$p=0$ for "dust," $p=(D-1)^{-1}\rho$ for "radiation," etc.].

Let us study now what are the effects of having n_c dimensions contracting. We shall assume that this contraction happens in an adiabatic way, i.e., that

$$\nabla_m S_D{}^m = 0, \tag{5}$$

where $S_D{}^m$, the entropy D-vector, is given in terms of the mean four-velocity by $S_D{}^m = \boldsymbol{S}_D \cdot u^m$. The entropy density will then obey the relation

$$\boldsymbol{S}_D R^3 A^{n_c} \equiv X = \text{const}, \tag{6}$$

which implies that the total entropy in a unit of comoving volume is constant.

Let us first assume that the D-dimensional universe is uniformly filled with radiation (a similar analysis can be done for dust or other matter). Since the momenta associated with the extra dimensions will take on only discrete values, $q_i = 2\pi m_i/AL$, then the entropy density for this D-dimensional photon gas will be (in the local inertial frame)

$$\boldsymbol{S}_D(t) = -\frac{n_c+2}{(2\pi)^3(LA)^{n_c}}\, k \sum_{\vec{\mathrm{m}}} \int d^3p \{f_D \ln f_D - (1+f_D)\ln(1+f_D)\}, \tag{7}$$

where f_D is Planck's distribution function, which in this frame reads

$$f_D = \left(\exp\{\beta[p^2 + (2\pi/LA)^2 \vec{\mathrm{m}}^2]^{1/2}\} - 1\right)^{-1}. \tag{8}$$

In order to study the thermal evolution of the radiation gas, it is useful to consider two extreme periods: epoch I, when the extra dimensions are of an enormous size, so that $AL \gg \beta$, and epoch II, in which $AL \ll \beta$.

At epoch I, the discrete sum can be replaced by an integral, so that

$$S_D{}^{(\mathrm{I})} = c_D \beta^{-n_c-3}, \tag{9}$$

where

$$c_D = \frac{k(n_c+2)!\,\varphi(n_c+4)}{\pi^{n_c/2} 2^{n_c+2}\,\Gamma((n_c+3/2))}.$$

For a radiation-dominated universe ($\rho \sim R^{-3(4+n_c)/(3+n_c)} A^{-n_c(4+n_c)/(3+n_c)}$) the D-dimensional adiabaticity condition implies

$$T^{(\mathrm{I})} \sim R^{-3/(3+n_c)} A^{-n_c(3+n_c)}. \tag{10}$$

All the intrinsic D-dimensional quantities must now be related to some four-dimensional ones. It should be stressed that this is an artificial procedure during epoch I, and it is only justified by the fact that we know that after some time (namely, during epoch II) the four-dimensional universe will be singled out.

The D-dimensional distribution function is normalized via the particle density; i.e., the number of particles in a $D-1$ dimensional hypersurface V_{3+n_c} is given by

$$N(V_{3+n_c}) = \int f_D(\boldsymbol{p}, \boldsymbol{q}, t) R^3 d^3x A^{n_c} d^{n_c}y\, R^3 d^3p\, A^{n_c} d^{n_c}q. \tag{11}$$

This number can also be obtained in a formal way using $V_{3+n_c} = V_x \times V_y$, where V_x is a three-dimensional hypersurface and V_y an n_c-dimensional one. For an observer not sensitive to the extra dimensions, the probability of finding a particle in V_x with momentum between p and $\boldsymbol{p}+d\boldsymbol{p}$ at time t is intuitively given by integrating out f_D over the extra degrees of freedom, namely y and $\boldsymbol{q}$. In fact, the equivalent four-dimensional distribution function $f_4(p, t)$ can be defined in a natural way by

$$f_4(p, t) \equiv \frac{\int f_D A^{n_c} d^{n_c}q}{\int A^{n_c} d^{n_c}\boldsymbol{q}} \equiv \frac{\int f_D\, d^{n_c}q}{V(q)}, \tag{12}$$

so that

$$\int f_4 R^3 d^3x\, R^3 d^3p = \frac{N(V_{3+n_c})}{A^{n_c} V(q) \int A^{n_c} d^{n_c}y} \equiv \frac{N(V_{3+n_c})}{A^{n_c} V(q) A^{n_c} V(y)}.$$

Both $V(q)$ and $V(y)$ are formally infinite, reflecting the fact that we have normalized in D dimensions to a finite quantity. They will eventually disappear though in all ratios of physical quantities computed with f_4. In the second epoch (when $A \to 0$), we recover the natural result

$$f_4(\boldsymbol{p}, t) = f_D(\boldsymbol{p}, A=0, t).$$

Explicit computation of f_4 in epoch I, using (12) and (8) gives

$$f_4 = V(q)^{-1}\pi^{(n_c-1)/2}2^{(n_c+1)/2}A^{-n_c}\beta^{(1-n_c)/2}(Rp)^{(n_c+1)/2}\sum_{m=1}^{\infty} m^{(1-n_c)/2}K_{(1+n_c)/2}(\beta m R p), \tag{13}$$

whose infrared limit is

$$f_4 \sim V(q)^{-1}2^{n_c}\pi^{(n_c-1)/2}[(n_c-1)/2]!(\beta A)^{-n_c}\varphi(n_c), \tag{14}$$

and which in the ultraviolet behaves as

$$f_4 \sim V(q)^{-1}2^{n_c/2}\pi^{n_c/2}A^{-n_c}\beta^{-n_c/2}(Rp)^{n_c/2}e^{-\beta Rp}. \tag{15}$$

These limit behaviors are similar to the ones for the four-dimensional Planck function; the main differences are that the zero in the ultraviolet gets some power corrections, and that the infrared behavior is regulated by an effective cutoff.

In view of the complicated exact expression for f_4, it seems only natural to represent its physical effects by a Planck function with an effective temperature determined in such a way that the mean energy of this fictitious Planck gas is the same as the one stemming from our f_4; i.e.,[6]

$$T_{\text{eff}} = \left[\frac{2^{n_c+2}}{3V_q}\pi^{(n_c-1)/2}\frac{\varphi(n_c+4)}{\varphi(4)}\Gamma\left(\frac{n_c+5}{2}\right)\right]^{1/4}A^{-n_c/4}T^{(n_c+4)/4}. \tag{16}$$

The corresponding effective four-dimensional entropy satisfies

$$(S_4R^3)_J \sim R^{3n_c/(3+n_c)}A^{-(3n_c/4)(2n_c+7)/(n_c+3)}. \tag{17}$$

This is the most important result of our paper. It implies that the whole process produces entropy in the four-dimensional world, which would be considerable if A decreases very rapidly, and/or R grows very fast, and/or the number of extra dimensions is quite large.

This approximation ($A\to\infty$, epoch I) is valid from the "big bang" on, up to a critical time, to be determined by the condition

$$AL = \beta_0 R^{3/(3+n_c)}A^{n_c/(3+n_c)}. \tag{18}$$

At epoch II only the $\vec{m}=0$ mode in the sum contributes, so that

$$S_D^{\text{II}} = \tfrac{1}{2}(n_c+2)S_4^{\text{II}}(LA)^{-n_c}. \tag{19}$$

This implies that from this critical time on, the expansion is adiabatic in the four-dimensional sense also,

$$(S_4R^3)_{\text{II}} = \frac{2L^{n_c}}{n_c+2}X, \tag{20}$$

so that all the effects of the extra dimensions on the four-dimensional physics have disappeared.

It is not difficult to find solutions with the desired general characteristics. Starting from Einstein's equations, which in our case reduce to

$$\begin{aligned}
&3\left(\frac{\dot R}{R}\right)^2 + 3n_c\frac{\dot R}{R}\frac{\dot A}{A} + \frac{n_c}{2}(n_c-1)\left(\frac{\dot A}{A}\right)^2 = \lambda + 8\pi G\rho,\\
&2\frac{\ddot R}{R} + \left(\frac{\dot R}{R}\right)^2 + 2n_c\frac{\dot R}{R}\frac{\dot A}{A} + n_c\frac{\ddot A}{A} + \frac{n_c}{2}(n_c-1)\left(\frac{\dot A}{A}\right)^2 = \lambda - 8\pi G p,\\
&(n_c-1)\frac{\ddot A}{A} + \frac{(n_c-1)(n_c-2)}{2}\left(\frac{\dot A}{A}\right)^2 + 3(n_c-1)\frac{\dot R}{R}\frac{\dot A}{A} + 3\frac{\ddot R}{R} + 3\left(\frac{\dot R}{R}\right)^2 = \lambda - 8\pi G p,
\end{aligned} \tag{21}$$

it is very easy to check that when $n_c = 1$ a vacuum solution exists with zero cosmological constant, and such that $R\sim t^{1/2}\sim A^{-1}$ (this, however, is too slow a rate to be of any help with the horizon and flatness problems).

For both dust ($p=0$) and radiation in the large-n_c limit there is a solution of the type

$$A\sim R^{-m},$$
$$R\sim[R_0^{mn_c/2} - (4\pi G\rho_0)^{1/2}t]^{-2/mn_c}. \tag{22}$$

The behavior $A \sim R^{-m}$ is not strictly compatible with the compactification of the extra dimensions, as this *Ansatz* is valid only during epoch I. On the other hand, in epoch II the extra dimensions essentially decouple, so that the universe is well described by the standard four-dimensional model. A really satisfactory solution should incorporate in an essential way the fact that C_{n_c} is a compact manifold, even during epoch I.

In an universe evolving in accordance with a scenario of this type the amount of entropy produced can be as large as $\Delta(S_4 R^3) = 10^{86}$ (which is the amount necessary to solve the horizon and flatness problems, see Guth[4]) and the Hubble constant H_0 can be fixed to be equal to the actual observed value of 50 km/sec Mpc $\sim 10^{-18}$ sec^{-1}. Of course this implies neglect of the duration of epoch II, which is very short in this model. (The universe is then much older than 10^{10} years!) It cannot be taken seriously as representing our universe though, because, for example, it predicts too large a variation with time of the fine-structure constant by many orders of magnitude (it is known observationally to be $\dot{\alpha} < 10^{-24}$ sec^{-1}, see Dyson[7]). Nevertheless, we think that its existence makes it plausible that a solution of Einstein's equations will exist with all the desired characteristics.

We want to stress, finally, that the fact that the effective coupling constants in four dimensions cannot have too large a variation with time is a major constraint in all Kaluza-Klein cosmologies. It implies that after a period of more or less quick contraction the extra dimensions must remain almost stationary—with a very small radius, presumably smaller than $10^{17} m_p^{-1}$ (see Ref. 1) —until now. It is amusing to remark, in this connection, that it is not possible for either vacuum, dust, or radiation, within the context of the given *Ansatz* (1) for the metric, to build a model in which the ordinary dimensions, independently of their number, expand and the extra ones remain strictly stationary. (It seems possible to have $\dot{A}$ small in our vicinity—for $z < 1000$, say—in such a way that $\dot{\alpha}$ is below the experimental limit; but dust or radiation models in which $\dot{A} = 0$ but $\dot{R} \neq 0$ must necessarily have a more complicated topology than the one assumed here.)

In conclusion, we have suggested a cosmological scenario for the production of large amounts of entropy in an universe undergoing dynamical compactification of its extra dimensions. The possibility of solving the horizon and flatness problems in this type of scenario is of course very exciting. Further research on these topics is currently under way.

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[1]See, for example, A. Salam and J. Strahdee, Ann. Phys. (N.Y.) 141, 316 (1982).

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[4]A. Guth, Phys. Rev. D 23, 347 (1981).

[5]We are adopting here the kinetic viewpoint that the dimensions of the momentum in $4 + n_c$ dimensions are the same as in ordinary space-time, namely 1 in natural units. In other words, the "classical" definition $p^\mu \equiv m\, dx^\mu/ds$ is adopted.

[6]Other definitions are possible; one could, for example, define the effective temperature in such a way that the particle density is the same as the one stemming from f_4, i.e.,

$$T_{\text{eff}}' = \left[\frac{\pi^{(n_c+1)/2}}{4V_q} \frac{\varphi(n_c+3)}{\varphi(3)} \frac{\Gamma(n_c+3)}{\Gamma((n_c+3)/2)} \right]^{1/3} \times A^{-n_c/3} T^{(n_c+3)/3}.$$

If the original distribution function, f_4, is not very far from equilibrium all the possible effective temperatures, although different, should give roughly equivalent physical results. That this is the case for our f_4 can be easily checked by a direct comparison.

[7]F. J. Dyson, *Aspects of Quantum Theory*, edited by A. Salam and E. P. Wigner (Cambridge Univ. Press, Cambridge, England, 1972).