Kaluza-Klein Monopole

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By adding the trivial term $-dt^2$ one can convert the four-dimensional (positive-definite) Newman-Unti-Tamburino line element into a static solution of the five-dimensional vaccuum Einstein equations. Interpreted *à la* Kaluza and Klein, the solution describes a monopole with a charge-to-mass ratio of $\sqrt{2}$ in rationalized Planck units. Although it is a perfectly regular five-geometry it appears singular from a four-dimensional perspective.

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Present-day physics exhibits two independent kinds of symmetries: the geometrical symmetries expressing the interchangeability of the points of space-time, and the internal symmetries expressing an interchangeability among certain of the material structures which can occur at a single space-time point. The aim of the so-called Kaluza-Klein theories is to unify these types of symmetry by hypothesizing that the apparently internal symmetries are in reality also geometrical. One posits the existence of extra (but very compact) dimensions of space-time and interprets gauge transformations as generalized rotations taking place within these extra dimensions.

In the prototype Kaluza-Klein theory[1] the extra dimension was no more than a mathematical artifice since the internal metric (that of a circle) was fixed *a priori*, and not to be varied in the action principle. When this restriction is dropped excitations of the internal geometry become possible, and the extra dimensions can have observable effects.

Nevertheless a cynic might still choose to regard the use of such dimensions as a trick for producing interesting four-Lagrangians. Short of direct exploration of the "internal" geometry, the best antidote to such cynicism might be the discovery of a particle whose hypothesized structure *does not admit* of interpretation in terms of four-dimensional fields. In fact such structures exist, and we will see below that one which occurs in the purely electromagnetic theory can be interpreted as a magnetic monopole carrying the Dirac charge.

The Lagrangian.—For the Abelian [U(1)] Kaluza-Klein theory, space-time is a five-manifold ${}^5M$ with metric g_{AB} of signature $-++++$. To descend to four dimensions one introduces a vector field K^A whose orbits are circles ($\exp 2\pi K = 1$) and assumes that, at least to a first approximation, K is a Killing vector for g_{AB}. The five-metric then decomposes as

$$g_{AB}=\gamma_{AB}+A_A A_B\,, \tag{1}$$

where γ_{AB} is a degenerate metric of signature $0-+++$ such that $\gamma_{AB}K^B=0$. The one-form A is well defined wherever $K\neq 0$ and related to K^A by $A_A=K_A/\lambda=g_{AB}K^B/\lambda$, where $\lambda=(g_{AB}K^AK^B)^{1/2}$ is the radius of the "internal circle."

The effective four-manifold ${}^4M$ is now the *quotient* of ${}^5M$ with respect to the action of K: Each circular orbit of K becomes a single point in ${}^4M$. Because of the assumption that K is a symmetry, nothing really depends on the fifth dimension; and g_{AB} is equivalent to a set of three distinct fields defined on ${}^4M$, a scalar, a tensor, and a vector, deriving respectively from λ, γ_{AB}, and A_A. At spatial infinity λ will approach a constant value λ_∞, and we take the four-scalar field to be

$$\omega=-\tfrac{1}{2}\ln(\lambda/\lambda_\infty)\,. \tag{2}$$

To define the four-metric $\tilde{g}_{\mu\nu}$ (using Greek indices for ${}^4M$) notice that there is a unique metric $\gamma_{\mu\nu}$ whose pullback to ${}^5M$ is γ_{AB} and put

$$\tilde{g}_{\mu\nu}=(\lambda/\lambda_\infty)\gamma_{\mu\nu}=e^{-2\omega}\,\gamma_{\mu\nu}\,. \tag{3}$$

In a coordinate system with x^5 parametrizing the internal dimension and $K^A=(0,0,0,0,1)$ the components of g_{AB} are independent of x^5 and our definitions so far read $\lambda=(g_{55})^{1/2}$, $\tilde{g}_{\mu\nu}=(\lambda/\lambda_\infty)[g_{\mu\nu}-g_{\mu5}g_{\nu5}/g_{55}]$. Finally we will need the vector potential $\tilde{A}_\mu\equiv A_\mu/\lambda$. Unlike ω and $\tilde{g}_{\mu\nu}$, $\tilde{A}_\mu$ is gauge dependent—it changes under coordinate transformations $x^5\to x^5+\theta(x^\mu)$—but of course

$$\tilde{F}_{\mu\nu}\equiv(\mathrm{curl}\tilde{A})_{\mu\nu}\equiv\partial_\mu\tilde{A}_\nu-\partial_\nu\tilde{A}_\mu \tag{4}$$

is not, and can be defined invariantly as the two-form on ${}^4M$ whose pullback to ${}^5M$ is $F\equiv\mathrm{curl}(A/\lambda)$.

With these definitions[2] the five-Lagrangian

$(2k)^{-1}({}^5R)(-{}^5g)^{1/2}$ leads to a four-action

$$S = S_\infty + \kappa^{-1}\int(\tfrac{1}{2}\tilde{R} - 3|\tilde{\nabla}\omega|^2 - \tilde{\nabla}^2\omega - e^{-6\omega}\lambda_\infty{}^2/8\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})d\tilde{V}, \qquad (5)$$

where $\tilde{R}$ is the scalar curvature of $\tilde{g}_{\mu\nu}$ (with respect to which all contractions are done), and $\kappa = k/2\pi\lambda_\infty$ plays the role of the four-dimensional gravitational constant $8\pi G$. The integral S_∞ at spatial infinity would be needed in deducing the total energy from first principles, but we may neglect it for what follows.

A monopole solution.—Since the five-Lagrangian is just the scalar curvature the five-dimensional field equations are just ${}^5R_{AB}=0$. Consequently any signature-(++++) solution g_{AB} of four-gravity can be extended to a static Kaluza-Klein solution, g_{AB}, simply by setting $g_{00}=-1$, $g_{a0}=0$ (where $a, b = 1, 2, 3, 5$). For example the flat metric derived from $g_{ab}=\delta_{ab}$ by periodically identifying $x^5 = x^5 + 2\pi$ gives rise in this way to the Kaluza-Klein equivalent of Minkowski spacetime.

Applying this procedure to the so-called Taub-Newman-Unti-Tamburino (NUT) instanton yields a five-metric which can be written conveniently[3] in terms of Cartesian coordinates t, x, y, z, w for R^5. In terms of $q^2 = x^2+y^2+z^2+w^2$, $\rho = 2\ln q$, and $\sigma_z = 2q^{-2}(zdw - wdz + xdy - ydx)$ (with cyclic permutations for σ_x and s_y), the full Kaluza-Klein metric is

$$ds^2 = -dt^2 + U^{-1}d\rho^2 + 4m^2U\sigma_z{}^2 + (\rho^2 - m^2)(\sigma_x{}^2 + \sigma_y{}^2), \qquad (6)$$

where $U = (\rho - m)/(\rho + m)$. In (6) there is an apparent singularity at $\rho = m$. However, a simple radial coordinate transformation and redefinition of ρ brings ds^2 to a form valid for all t, x, y, z, w, and in particular shows that (for any $m > 0$) the topology of the solution is simply R^5. The metric g_{AB} thereby assumes the form[4]

$$-\partial t^2 + \left(1 + \frac{m}{\rho}\right)^2 \partial r^2 + \left(\frac{2mr}{\rho + m}\right)^2 \sigma_z{}^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2), \qquad (7)$$

where $r \equiv x^2+y^2+z^2+w^2$ and

$$\rho^2 \equiv r^2 + m^2. \qquad (8)$$

An equivalent expression in terms of Euler angles for the three-sphere can be obtained by substituting[3]

$$\sigma_z = -\partial\psi - \cos\theta\,\partial\varphi, \qquad (9)$$

$$\sigma_x{}^2 + \sigma_y{}^2 = \partial\theta^2 + \sin^2\theta\,\partial\varphi^2 \equiv \partial\Omega^2, \qquad (10)$$

with $0 \le \theta \le \pi$ and with φ and ψ having periods 2π and 4π, respectively.[5] That (7) is regular at $r=0$ may not be immediately recognizable, but direct computation reveals that near $r=0$ it reduces to $-\partial t^2 + 4(\partial w^2 + \partial x^2 + \partial y^2 + \partial z^2)$, whose smoothness is manifest.

Equation (7) defines the five-dimensional soliton solution. It is manifestly static, already in the form (1) with $A = 2mr/(\rho + m)\sigma_z$, and in $t, r, \theta, \varphi, \psi$ coordinates clearly admits $K = 2\,\partial/\partial\psi$ as Killing vector with period 2π (i.e., $\psi/2$ can be used to parametrize the internal circles[6]). It follows that

$$\lambda = 4mr/(\rho + m). \qquad (11)$$

For $r \gg m$ the metric approaches $-\partial t^2 + \partial r^2 + r^2\partial\Omega^2 + 4m^2\sigma_z{}^2$, which locally looks like a "Kaluza-Klein Minkowski space" with internal radius $8\pi m$; in particular $\lambda_\infty = 4m$.

Proceeding from these formulas it is easy to pass to the four-description of the metric. The quotient manifold ${}^4M$ is naturally coordinatized by t, r, θ, φ and is thus topologically R^4. The true space-time ${}^5M$ is everywhere a fiber bundle over ${}^4M$ *except at the origin* $r=0$, where $\lambda = 0$ and the U(1) fiber shrinks to a point. From (10) $\gamma_{\mu\nu}$ is $-\partial t^2 + (1 + m/\rho)^2\partial r^2 + r^2\partial\Omega^2$, whence the quotient metric $\tilde{g}_{\mu\nu}$ of (3) is [with the help of (8)]

$$-[(\rho - m)/r]\,\partial t^2 + (r/\rho)(1 + m/\rho)\partial r^2 + (\rho - m)r\,\partial\Omega^2. \qquad (12)$$

From (2) and (11) we see that

$$\omega = \tfrac{1}{2}\ln[(\rho + m)/r]. \qquad (13)$$

Finally $\mathrm{curl}(A/\lambda) = \frac{1}{2}\mathrm{curl}\,\sigma_z = \frac{1}{2}(\sin\theta\,\partial\theta \wedge \partial\varphi)$ which translates immediately to ${}^4M$ to give for the electromagnetic field tensor

$$\tilde{F} = \tfrac{1}{2}\sin\theta\,\partial\theta \wedge \partial\varphi. \qquad (14)$$

Observe that $\tilde{g}_{\mu\nu}$, ω, and $\tilde{F}$ all become singular at $r=0$, as could have been anticipated from the breakdown there of the local product structure of ${}^5M$.

Mass and charges of the solution.—To justify calling this solution a monopole let us compute its electromagnetic charges. The total magnetic flux through any sphere about the origin is, from (14),

$$\oint \tilde{B}\cdot ds = \oint \tfrac{1}{2}\tilde{F}_{\mu\nu}\,d\Sigma^{\mu\nu} = \int \tfrac{1}{2}\sin\theta\,d\theta\,d\varphi = 2\pi,$$

as could have been inferred directly from the fact that the NUT metric has one twist[7] when regarded as a U(1) bundle over ${}^4M$. However, this flux is not immediately the magnetic charge because the coefficient of $F_{\mu\nu}F^{\mu\nu}$ in (5) differs from the customary $-\frac{1}{4}$, the correctly normalized magnetic charge[8] being rather $Q_B = 2\pi\lambda_\infty \times (2\kappa)^{-1/2}$. In units such that $k = 2\pi\lambda_\infty = 1$ (note, $\hbar \neq 1$!), which I designate "KK units," this becomes $Q_B = 1/\sqrt{2}$.

As for the total *electric* flux, $\oint \tilde{E} \cdot dS$, it vanishes because the absence of ∂t in (14) means that E itself vanishes. Notice incidentally that the Dirac condition $eQ_B/h =$ integer (where $h = 2\pi\hbar$) is verified with $n = 1$ since the unit of electric charge is $e = (2\kappa)^{1/2}\hbar/\lambda_\infty$.

The monopole's scalar charge is somewhat easier to compute. According to (13) and (8) $\omega \approx m/2r$ at large radii and is therefore the field of a source of strength $2\pi m = (\pi/2)\lambda_\infty$. Again, however, ω should be rescaled by $(6/\kappa)^{1/2}$ to bring its contribution to (5) to the usual form. The resulting scalar charge is $Q_S = (3/2\kappa)^{1/2}\pi\lambda_\infty$ or $(\frac{3}{8})^{1/2}$ in KK units.

Finally, what about the mass? With the definitions of this paper we can deduce the total energy M unambiguously[9] from the behavior of $\tilde{g}_{00}$ as $r \to \infty$. From (12) $-\tilde{g}_{00} \sim 1 - m/r$, whence $M = 4\pi m/\lambda_\infty = \pi\lambda_\infty/\kappa$ or $\frac{1}{2}$ in KK units. To summarize, then, the mass and charges in KK units are[10]

$$M = \tfrac{1}{2}, \quad Q_B = (\tfrac{1}{2})^{1/2}, \quad Q_S = (\tfrac{3}{8})^{1/2}.$$

Incidentally the value of Q_B implies that bound states uniting the monopole with charge e excitations of g_{AB} would be fermions.[11]

Discussion.—In no four-dimensional theory with a scaling symmetry can there be a stationary solution of nonzero energy.[12] How then do the Kaluza-Klein equations admit a solution (7) which is not only stationary, but everywhere regular and without event horizon? Two features seem essential. The inescapable five-dimensionality of the metric prevents any four-theorems from being applied directly; and the fixed value of the internal radius λ_∞ in effect introduces a fundamental length which then prevents the five-dimensional analog theorem from applying. This seems quite similar to the way in which a boundary condition of the Higgs field allows monopole solutions to persist in the Prasad-Sommerfield limit, even though the scale-invariance-breaking potential itself goes to zero. Here, however, there is no Higgs field and no symmetry breaking.

In the same way as we used the NUT metric, each of the known ALF instantons could be used to construct a static Kaluza-Klein solution. For example the multi-Taub-NUT solutions become multimonopole solutions and the Schwarzschild instanton of "mass" M yields a neutral solution of total energy $M/2$ which seems to describe a delicately balanced "bubble" of the type discussed by Witten.[13] By a slight generalization of the present procedures (to allow g_{0A} to vary) the Israel-Wilson solutions may also turn into five-dimensional vacuum solutions.

Of course all these solutions are physically unrealistic insofar as U(1) is not the gauge group realized in nature.[14] But in order that there be analogs it may suffice that the "correct" G contain U(1) as a subgroup, which is virtually certain.

In a way it is surprising that so many seemingly distinct particlelike structures can exist in what seem to be fundamental theories.[15] One might wish that at least some of these possibilities could be, if not excluded, then at least reduced to some of the others. In the context of the present theory, for example, there is no reason why the charge carried by the monopole should be labeled "magnetic." It might equally well be deemed "electric" and $\tilde{A}_\mu$ treated as a potential for the dual field $*F_{\mu\nu}$. With this reidentification— or even without it—it becomes important to understand how the effective dual potential needed to express the mutual interaction of such monopoles can arise. Once this is done,[16] it may appear that some or all of what today seem to be elementary structures (including space-time itself) arise in a similar way from "sub-Planck-scale" quantum processes.

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[1]Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Math. Phys. 1921, 966; O. Klein, Z. Phys. 37, 895 (1926).

[2]And with the following conventions: $c \equiv 1$, $[\nabla_M, \nabla_N]V^A = R^A{}_{BMN}V^B$, $R_{MN} = R^A{}_{MAN}$.

[3]The Taub-NUT metric is described by A. Taub and C. W. Misner, Zh. Eksp. Teor. Fiz. 55, 233 (1968) [Sov. Phys. JETP 28, 122 (1969)]. The present metric corresponds to the choice $l = im$ (with "ρ" denoting their coordinate t).

[4]Here "∂t" denotes the one-form $\partial_A t$; "∂t^2" stands

for $\partial t \otimes \partial t$, i.e., $\partial_A t\, \partial_B t$; and σ_x, σ_y, and σ_z are to be understood as one-forms defined in R^4.

[5]To remove the familiar polar coordinate singularities one makes the unfamiliar extra identifications at $\theta = \pm\pi$, $(\theta, \varphi, \psi) = (\theta, \varphi + \Delta, \psi \mp \Delta)$, Δ arbitrary.

[6]Except near $\theta = \pi$ where $\psi - \varphi$ can be used.

[7]S. Ramaswamy and A. Sen, J. Math. Phys. 22, 2621 (1981).

[8]In so-called Heaviside units for which the force between two charges is $Q_1 Q_2/4\pi r^2$.

[9]There can be only one conserved quantity whose "matter" contributions (from the stress-energy tensors for ω and $F_{\mu\nu}$) are correctly normalized in regions of small gravitational field. It was for this convenience in identifying M that I chose to use (3) and not just γ as the four-metric.

[10]Thus the present solution has an electromagnetic charge greater than that of any Reissner-Nordstrom metric of the same mass.

[11]See A. O. Barut, Phys. Rev. D 10, 2712 (1974); R. D. Sorkin, Phys. Rev. D 27, 1787 (1983).

[12]This follows from a theorem of B. F. Schutz and R. D. Sorkin, Ann. Phys. (N.Y.) 107, 1 (1977), with the assumption that the theory is defined by an "unconstrained" action principle.

[13]E. Witten, Nucl. Phys. B195, 481 (1982). Notice though that with the metric $\tilde{g}_{\mu\nu}$ the bubble becomes a point singularity and there is actually no "hole" in the apparent space-time.

[14]They are probably also quantum mechanically unstable via the mechanism of Ref. 13. However, one should at least decide which ones are classically stable, say with the methods of G. W. Gibbons and C. M. Hull, Phys. Lett. 109B, 190 (1982).

[15]For some examples see E. Lubkin, Ann. Phys. (N.Y.) 23, 233 (1963); R. D. Sorkin, J. Phys. A 10, 717 (1977), and 11, 795 (1978); D. Finkelstein, J. Math. Phys. 7, 1218 (1966).

[16]B. Hu, in *Proceedings of the Guangzhou Conference on Particle Physics* (Science Press, Beijing, 1980).