Phase Correlations in One-Dimensional Disordered Systems

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The correlation function of the phases of a quantum particle in a one-dimensional random potential is defined and calculated analytically and numerically. Away from eigenstates it decays exponentially with the phase correlation length equal to half of the localization length, while for the eigenstates it saturates at nonzero value.

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A number of recent papers[1-3] have focused on the problem of the one-dimensional random potential. It was assumed that there exists some length over which the phase of the wave function is completely randomized. This phase-randomization assumption contains two separate issues. The first is that the phase correlations decay to zero and the second is that the limiting probability distribution of the phases is uniform. The phase-randomization assumption was later discussed by Lambert and Thorpe[4] and by Stone, Allan, and Joannopoulos.[5] They demonstrated that the limiting probability distribution need not be uniform but did not study the correlations of the phases. In the present Letter we define a phase correlation function (PCF) $q_{\text{ph}}(s)$ as the sensitivity of the phase at some point to the change in phase at another point a distance s away, averaged over all possible positions of these two points with fixed separation, and prove that away from eigenstates it decays exponentially with correlation length equal to half of the localization length. For the eigenstates the phases are not randomized and PCF saturates at a nonzero value.

The random system is considered only at zero temperature; therefore the phase averaging due to phonon scattering is neglected and the problem is reduced to that of a single particle in a random potential (electron-electron interactions are also neglected). The corresponding Schrödinger equation in convenient units is

$$\Psi''(x) + [k^2 - V(x)]\Psi(x) = 0. \tag{1}$$

This equation can be locally solved at any point x_n. The local solution $\Psi_n(x) = \exp\{ik_n(x - x_n)\}$ satisfies two boundary conditions, $\Psi_n(x_n) = 1$ and $\Psi_n{}'(x_n) = ik_n$, where $k_n = [k^2 - V(x_n)]^{1/2}$.[6] The complex conjugate $\Psi_n{}^*(x)$ of the local solution $\Psi_n(x)$ is also a local solution of a Schrödinger equation. A general function, locally satisfying Eq. (1) at x_n, can be written as a linear combination of the two local solutions:

$$\Psi(x) = A_n \Psi_n(x) + B_n \Psi_n{}^*(x). \tag{2}$$

The coefficients at adjacent points are connected by the transfer matrix[7]

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \left(\frac{k_n}{k_{n+1}}\right)^{1/2} \begin{pmatrix} \exp(i\alpha_n)/\cos(h_n) & \exp(i\beta_n)\tan(h_n) \\ \exp(-i\beta_n)\tan(h_n) & \exp(-i\alpha_n)/\cos(h_n) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{3}$$

where h_n determines the transmission, while α_n and β_n determine the local phase shifts of the reflected and transmitted waves. Since the Schrödinger equation is linear it is sufficient to consider currentless (i.e., real) wave functions

$$\Psi(x) = A_n \Psi_n(x) + A_n{}^* \Psi_n{}^*(x). \tag{2a}$$

Complex coefficients A_n can be rewritten in the form

$$A_n = \tfrac{1}{2}\exp(G_n/2 - i\varphi_n/2), \tag{4}$$

where G_n and φ_n are real numbers.

The transfer matrix defined in Eq. (3) leads to the following relation for the G_n's[7-9]:

$$G_{n+1} = G_n + \ln\left(\frac{k_n}{k_{n+1}}\right) + \ln\left(\frac{1 + y_n{}^2}{\omega_{n-1} + y_n{}^2/\omega_{n-1}}\right), \tag{5}$$

where $k_n = [k^2 - V(x_n)]^{1/2}$, $\omega_n = \tan^2(\frac{1}{4}\pi - \frac{1}{2}h_n)$, and y_n can be determined from the recurrence relations

$$y_{n+1} = \omega_n(y_n - r_n)/(1 + y_n r_n), \tag{6}$$

with

$$r_n = \tan(\rho_n), \tag{7a}$$

where

$$\rho_n = \tfrac{1}{2}(\alpha_n - \beta_n + \alpha_{n-1} + \beta_{n-1}). \tag{7b}$$

The phases φ_n defined in Eq. (4) are related to y_n by

$$y_n = \tan(f_n), \tag{8a}$$

where

$$f_n = \tfrac{1}{2}(\varphi_n + \alpha_{n-1} + \beta_{n-1}). \tag{8b}$$

In this notation the recurrence relations [Eq. (6)] can be rewritten in the form

$$\tan(f_{n+1}) = \omega_n \tan(f_n - \rho_n) = \omega_n \tan(\bar{f}_n), \tag{9a}$$

where

$$\bar{f}_n = f_n - \rho_n = \tfrac{1}{2}(\varphi_n - \alpha_n + \beta_n). \tag{9b}$$

Substitution of Eqs. (8a) and (9a) into Eq. (5) leads to

$$G_{n+1} = \ln\left(\frac{k_{\text{out}}}{k_{n+1}}\right) + \sum_{m=1}^{n} \ln\left(\frac{\cos^2(\bar{f}_{m-1})}{\omega_{m-1}\cos^2 f_m}\right), \tag{10}$$

where k_{out} corresponds to the value of the potential outside the system.

Two independent solutions can be obtained by choosing $y_1 = \pm 1$, $\alpha_0 = k_0 = 0$, $\beta_0 = \pi/2$. The linear combination of these solutions that has no incoming wave at one end of the system gives the dimensionless Landauer resistance[8,10]

$$R_L = T_L^{-1} - 1 = \{\exp(G_L^+) + \exp(G_L^-) - 2\}/4, \tag{11}$$

where T_L is the transmission coefficient and superscripts (±) correspond to $y_1 = \pm 1$.

In a random system the absence of the long-range order means that the boundary effects $G_L^+ - G_L^-$ go to a constant as $L \to \infty$. Then $R_L \propto \exp(G_L) \propto \exp(2x_L/L_0)$, where $G_L = (G_L^+ + G_L^-)/2$ and $L_0 = \lim(2x_L/\ln R_L)$ is the wave-function localization length. This is confirmed in a numerical calculation for a variety of random potentials. The dependence of the resistance R_L on the length of the system for a potential $V(x) = \sum_{n=1}^{L} V_n \times \delta(x-n)$ is presented in Fig. 1 (solid line) for wave number $k = 1.7$ and the probability distribution of the random potential $P(V_n) = \frac{1}{2}\delta(V_n - 2) + \frac{1}{2}\delta(V_n)$.

By Eqs. (2a) and (4) resistance of a system of length x_L is proportional to the particle probability density $|\Psi(x_L)|^2$. In the analysis below we distinguish two qualitatively different behaviors of the particle probability density in the random system. Case I is the exponential decay of $|\Psi(x_L)|^2$, corresponding to the particle of some arbitrary energy, not equal to the eigenstate, reflecting from the system. The transmission coefficient in this case is exponentially small, $T_L \propto \exp(-2x_L/L_0)$, and the particle is "localized" outside the system (solid line in Fig. 1). Case II is the exponential localization of the particle at some point inside the system. The probability density first increases exponentially, peaks at the point of localization, and then exponentially decays (dashed line in Fig. 1). This case corresponds to the resonance tunneling through the eigenstates of the system.[7,8]

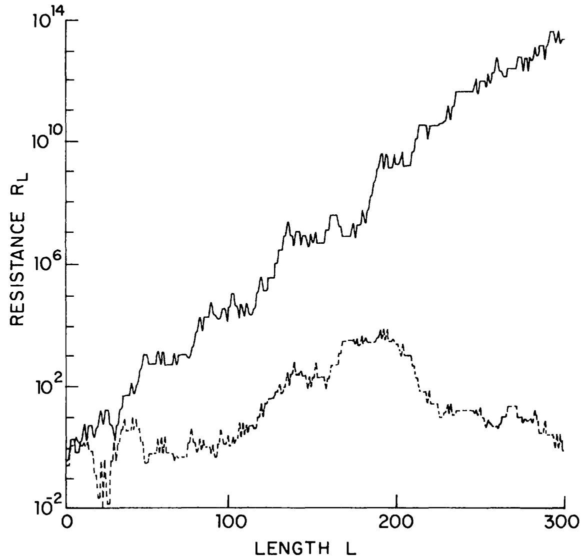

FIG. 1. The dependence of the resistance on the system length for the potential $V(x) = \Sigma_{n=1}^{L} V_n \delta(x-n)$ with the probability distribution $P[V_n] = \frac{1}{2}\delta(V_n - 2) + \frac{1}{2}\delta(V_n)$ and for wave vectors $k = 1.7$ (solid line) and the localized eigenstate $k = 1.975\,025\,22$ (dashed line) is calculated by exact Eqs. (1)–(11) for some random configuration. Away from eigenstates resistance increases exponentially with $L_0 \cong 18.9$, while at the eigenstate it peaks at the region of localization.

PCF is defined below and it is proven that its behavior is different in cases I and II. Let us introduce a small change $\delta\varphi_n$ of the phase φ_n at the point x_n. We are interested in the induced phase change at other points. Since the phases α_n and β_n in the transfer matrix [Eq. (3)] are fixed by the random potential, from Eqs. (7b), (8b), and (9b) it follows that ρ_n remains unchanged and the corresponding changes of f_n and $\bar{f}_n$ are $\delta f_n = \delta\bar{f}_n = \frac{1}{2}\delta\varphi_n$. The recurrence relation [Eq.

(9a)] can now be used to find the induced phase change at the adjacent point:

$$\delta\varphi_{n+1}=\omega_n(\cos^2 f_{n+1}/\cos^2\bar{f}_n)\,\delta\varphi_n\,. \tag{12}$$

Applying Eq. (12) p times we determine the induced phase change p points away:

$$\delta\varphi_{n+p}=\exp\left\{\sum_{m=n+1}^{n+p}\frac{\cos^2\bar{f}_{m-1}}{\omega_{m-1}\cos^2 f_m}\right\}\delta\varphi_n\,. \tag{13}$$

Combining Eq. (10) with Eq. (13) we find

$$\delta\varphi_{n+p}=(k_{n+1}/k_{n+p+1})\exp\{-(G_{n+p+1}-G_{n+1})\}\,\delta\varphi_n\,. \tag{14}$$

In what follows we assume that there is no overall potential gradient and that we can choose points x_n with the same value of random potential $V(x_n)=V_{\rm av}$, so that all k_n's are the same and their ratio drops out from Eq. (14).

Since the resistance of the segment is proportional to the exponential of G_L, the sensitivity of the phase at one point to the change in phase at another point is inversely proportional to the resistance of the segment of the system between these points:

$$\delta\varphi_{n+p}\propto\delta\varphi_n/R(x_n,x_{n+p})\,. \tag{15}$$

Resistance increases exponentially with length in case I (solid line in Fig. 1) and the phase φ_{n+p} at the point x_{n+p} in a given random system is independent of the phase φ_n at x_n with exponential accuracy and depends only on the local potential within L_0 of point x_{n+p}.

Suppose we start with two waves which are out of phase by π at point x_n; the phase difference at point x_{n+p} is a measure of the phase correlation between these points:

$$\bar{q}(x_n,x_{n+p})=\pi^{-1}[\varphi_{n+p}(\varphi_n{}^+)-\varphi_{n+p}(\varphi_n{}^-)]\,, \tag{16}$$

where $\varphi_{n+p}(\varphi_n{}^{\pm})$ is the phase at point x_{n+p} induced by two independent phases $\varphi_n{}^+=0$ and $\varphi_n{}^-=\pi$ at x_n. Since a wave function with arbitrary phase can be represented as a linear combination of two wave functions with phases 0 and π, the quantity $\bar{q}(x_n,x_{n+p})$ describes the phase correlations in a given random sequence between points x_n and x_{n+p}. From Eq. (15) it follows that

$$\bar{q}(x_n,x_{n+p})\propto 1/R(x_n,x_{n+p})\,, \tag{17}$$

and the extensive quantity which scales and has regular fluctuations $[\propto(x_{n+p}-x_n)^{-1/2}]$ is $\ln[\bar{q}(x_n,x_{n+p})]$. However, the phase correlation at separations $s=x_{n+p}-x_n$ is described by the average of the function itself, $q_{\rm ph}=\langle\bar{q}(x_n,x_{n+p})\rangle$, rather than its logarithm. We can consider either ensemble average or the average over all possible positions of two points with fixed separation s inside the system. PCF is similar to the spin-correlation function of random Ising model.[11]

For disorder on a lattice with unit spacing ($x_n=n$), PCF is defined as

$$q_{\rm ph}(s)=\frac{1}{L-s}\sum_{n=1}^{L-s}\bar{q}(n,n+s)\,. \tag{18}$$

In the case I $\bar{q}(n,n+s)\propto 1/R(n,n+s)\propto\exp(-2s/L_0)$ and PCF decays exponentially,

$$q_{\rm ph}(s)\propto\exp(-2s/L_0)\,. \tag{19}$$

The phase "forgets" its initial value exponentially fast. The numerical calculation (solid line in Fig. 2) verifies this result.

PCF exhibits more interesting behavior in case II. At the eigenenergy of a given random system the correlation between the phases at the end point and the point a distance s away first exponentially decays with s, reaches an exponentially deep minimum in the localization region, and then exponentially increases with s, and can even be of order unity if the localization point is near the center of the system. The correlations between the phases at the points with equal values of resistance (which are located at approximately[12] equal distances from the localization region)

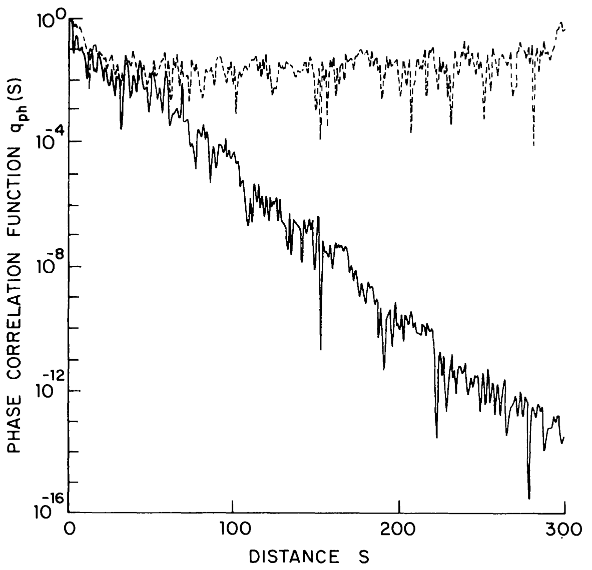

FIG. 2. The phase correlation function (PCF) for the systems introduced in Fig. 1 is calculated by Eq. (18). Away from eigenstates it decays exponentially with the phase correlation length approximately equal to half of the localization length (solid line). At the eigenstate PCF saturates (dashed line).

in a given system are of order unity, no matter how far off these points are from each other. The phase, which has exponentially well "forgotten" its initial value, starts "recalling" it after the localization point. In the sum of Eq. (18) most of the terms are proportional to $\exp(-2s/L_0)$ and only a small fraction ($\propto L_0/2L$) of them are of the order unity. Thus

$$q_{\mathrm{ph}}(s) \propto \exp(-2s/L_0) + L_0/2L . \tag{20}$$

As the size of the system increases ($L \to \infty$) Eq. (20) approaches Eq. (19). However, at large distances $s > (L_0/2)\ln(L/2L_0)$ the correlation saturates. In order to calculate the corrections to Eq. (20) one has to take into account the fluctuations of the resistance (Fig. 1) and boundary effects.

The results of the numerical calculation for the eigenstate $k = 1.975\,025\,22$ of the system of length $L = 300$ with potential $V(x) = \sum_{n=1}^{L} V_n \delta(x-n)$ and the probability distribution $P(V_n) = \frac{1}{2}\delta(V_n - 2) + \frac{1}{2}\delta(V_n)$ are presented in Fig. 2 (dashed line). The corresponding localization length is $L_0 \cong 20$ and PCF saturates for $s > 20$ [the theoretical value is $(L_0/2)\ln(L/2L_0) \cong 20.1$] at the value $q_{\mathrm{ph}} \cong \frac{1}{30}$ predicted by the theory. Large fluctuations are due to the fact that $\bar{q}$ is not an extensive quantity and is not averaging well. The average of $\ln\bar{q}$ is a smooth curve. This problem is similar to the one with average and representative resistances. The fluctuations are increasing with s and the slope of the saturation line becomes positive because we are averaging over smaller number of pairs for larger separations.

Summarizing, we would like to stress that Eq. (20) and Fig. 2 (dashed line) imply that there are finite correlations of eigenstate phases at large separations in random system. The phases of the eigenstates are not randomized. Away from eigenstates the phase correlation length is equal to half of the localization length.

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