Baryons as Chiral Solitons

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Topologically stable solitonic solutions to Skyrme's chiral-invariant Lagrangian have been obtained numerically. The single parameter in these solutions has been determined with the Goldberger-Treiman relation. The identification of these objects with baryons leads to sensible statements regarding the sizes and energies of baryons as well as the two- and three-body interactions between baryons at zero separation.

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Twenty years ago Skyrme[1] proposed a model in which low-lying baryons emerge from a nonlinear meson field theory as topologically twisted configurations or solitons. The topological charge or winding number was identified with the baryon charge. This model has recently been revived[2,3] in the context of quantum chromodynamics (QCD). In particular, Witten[3] has established an intriguing connection between Skyrme's soliton and baryons in large-N_C (where N_C is the number of colors) QCD.

Suppose that large-N_C QCD confines quarks. Then, in this limit,[4] QCD may be approximated at low energies by a weakly coupled field theory of mesons described by effective local fields with effective local interactions. Weakly coupled field theories sometimes possess solitonlike states. Witten has argued[4] that, in the large-N_C limit, baryon masses are proportional to N_C when viewed in terms of the planar diagrams of QCD or to f_π^2 when analyzed in an effective field theory (e.g., a nonlinear chiral Lagrangian). Thus, the identification of Skyrme solitons with baryons seems consistent with QCD, at least a low energies. That the Skyrme soliton is indeed a fermion has been proven by several authors.[3,5] In particular, the connection between the topological charge and the baryon charge has been shown to be unique with the number of colors, N_C, playing a crucial role.[3] Further progress in this direction is reported in the preceding Letter,[6] where it is shown that the baryon charge carried by the soliton field uniquely determines the way to marry the Skyrme soliton with the phenomenologically successful quark-bag picture.

The purpose of this note is to explore the possibility that the Skyrme soliton provides a satisfactory description of low-lying baryons and to see how far one can go in understanding low-energy nucleon and nuclear dynamics without invoking explicit quark degrees of freedom. For this, we will consider the Skyrme Lagrangian appropriate for the chiral SU(2)⊗SU(2) group,

$$\mathcal{L} = -\tfrac{1}{4}f_\pi^2\,\mathrm{Tr}(L_\mu L_\mu) - \tfrac{1}{4}\epsilon^2\,\mathrm{Tr}\{[L_\mu, L_\nu]\}^2;$$
$$L_\mu = U^\dagger \partial_\mu U, \tag{1}$$
$$U = f_\pi^{-1}[\sigma(x) + i\vec{\tau}\cdot\vec{\pi}(x)], \quad U^\dagger U = 1,$$

where f_π is the pion decay constant, ϵ is a constant to be determined, σ is the scalar meson field, and $\vec{\pi}$ is the triplet Goldstone boson (i.e., pion) field. Although the quadratic term in Eq. (1) cannot support a topological soliton by itself for spatial dimension three, there are solitons[1,2] when a quartic term is added as in Eq. (1). Following Skyrme, we make the *hedgehog Ansatz*,

$$U(r) = \exp[i\vec{\tau}\cdot\hat{r}\,\theta(r)].$$

This corresponds to a mapping of the real-space three-sphere (S^3) onto the internal-symmetry three-sphere (S^3) for a configuration U which approaches a constant at spatial infinity. This mapping represents the third homotopy group $\pi_3(S^3) \sim Z$, the group of additive integers, labeled by an integer which is identified with the baryon number B.[1,3,5,6] The large-N_C expansion of QCD suggests that baryons should emerge as solutions to the classical equation of motion of Eq. (1) and the low-energy theorems of current algebra from tree diagrams for the same Lagrangian when appropriately fluctuated around the (soliton) background field.

It is convenient to write the energy of the soliton corresponding to the Lagrangian of Eq. (1) in terms of the variable τ defined as $\ln(r/r_0)$, where r_0 is an arbitrary (and redundant) scale factor. This yields

$$E = 2\pi f_\pi^2 r_0 \int_{-\infty}^{+\infty} d\tau\, e^{\tau}(\dot{\theta}^2 + 2\sin^2\theta) + (32\pi\epsilon^2/r_0)\int_{-\infty}^{+\infty} d\tau\, e^{-\tau}\sin^2\theta(\dot{\theta}^2 + \tfrac{1}{2}\sin^2\theta). \tag{2}$$

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The chiral angle, $\theta(\tau)$, may be found by minimizing the energy of the soliton. This leads to the Euler equation

$$\ddot{\theta}(1+Ae^{-2\tau}\sin^2\theta)+\dot{\theta}(1-Ae^{-2\tau}\sin^2\theta)-\sin 2\theta\{1-\tfrac{1}{2}Ae^{-2\tau}(\dot{\theta}^2-\sin^2\theta)\}=0, \tag{3}$$

where we have introduced the quantity A defined as $16\epsilon^2/f_\pi^2 r_0^2$. In the limit of large positive τ, Eq. (3) indicates that $\theta(\tau)$ equals $a\exp(-2\tau)$. This asymptotic form is also obtained for the ordinary hedgehog in which ϵ^2 and A are zero.[7] In the limit of large negative τ, $\theta(\tau)$ equals $B\pi+\alpha\exp\tau$ where B is an integer which can be identified with the baryon number. (The parameters a and α are to be determined by the solution of the Euler equation.) In this limit the ordinary hedgehog leads to a chiral angle diverging as $\exp(-\tau)$. This divergence is related to the instability of the ordinary hedgehog with respect to scale transformations.[7,8]

The explicit τ dependence of Eq. (3) indicates that solutions for different values of A are related by a trivial displacement in τ:

$$\theta_A(\tau)=\theta_{A_0}(\tau-\tfrac{1}{2}\ln(A/A_0)). \tag{4}$$

This leads to considerable numerical simplification. Equation (4) also permits us to rewrite the energy as

$$E=8\pi f_\pi|\epsilon|\{A_0^{-1/2}I_1^{A_0}+A_0^{1/2}I_2^{A_0}\}, \tag{5}$$

where the terms in Eq. (5) correspond to the terms in Eq. (2). The term in brackets is independent of A_0. Note also that Eq. (5) does not depend on r_0. However, by regarding Eq. (2) as a function of r_0 for fixed ϵ_2, it is apparent that the two terms in the brackets of Eq. (5) must be equal. This is the content of Derrick's virial theorem.[8] This provides an appealing test of numerical results: The difference between these terms is linear in the difference between an approximate $\theta(\tau)$ and the exact function; the energy is quadratic in this difference.

Equation (3) is well suited to numerical solution through the replacement of $\dot{\theta}$ and $\ddot{\theta}$ by finite-difference expressions. Initial values of the asymptotic parameters a and α were guessed, and the resulting quadratic equation solved to integrate Eq. (3) in (or out) to the point at which θ equals $(B-\frac{1}{2})\pi$. The parameters a and α were adjusted to yield continuity in θ and $\dot{\theta}$ at this point. Sufficient care was taken to ensure the equality of the terms in Eq. (5) to 0.2% (with substantially greater precision in their sum as noted above). From the resulting monotonic behavior of both the interior and the exterior derivatives of θ at the matching point, the solution for a given integer, B, would appear to be unique. This procedure is extremely simple and solutions were readily obtained with a pocket calculator. For $B=1$ the relevant parameters are $a=80.8$ and $\alpha=-0.328$ for $A_0=75$. Results for $B=1$–3 are given in Table I.

In dependent knowledge of ϵ^2 (e.g., from $\pi\pi$ scattering) would enable us to determine the energy and size of the Skyrme soliton for each value of B. Instead, we employ the Goldberger-Treiman relation which relates the chiral angle, $\theta(r)$, and the pion field for large r. This relation, familiar from the ordinary hedgehog,[7] is unaltered by the additional quartic term in Eq. (1) and assumes the form

$$\frac{16\epsilon^2 a}{f_\pi^2 A_0 r^2}=\frac{f_{\rm hh}}{4\pi\mu_\pi^2(f_\pi/\mu_\pi)r^2}. \tag{6}$$

Using the values $f_\pi=93$ MeV, $\mu_\pi=140$ MeV, $f_{\rm hh}=1.8$, and the value $A_0/a=0.928$ obtained from our $B=1$ solution, we find $\epsilon^2=0.005\,52$ which is in reasonable agreement with the limits obtained from $\pi\pi$ scattering.[9] This value of ϵ^2 was used to obtain the energies in Table I. Effects of the small empirical uncertainty in f_π (0.1%) can be estimated from Eqs. (5) and (6) but are far small-

TABLE I. Energies and sizes of Skyrme solitons with $B=1$–3 for $\epsilon^2=0.005\,52$. The integrals of Eq. (5) are also shown as a check of numerical accuracy.

	$r_{\rm rms}{}^B$ (fm)	$I_1^{A_0}/A_0^{1/2}$	$A_0^{1/2}I_2^{A_0}$	E^B (GeV)	$E^B/E^{B=1}$
$B=1$	0.48	4.1009	4.1031	1.42	1
$B=2$	0.68	12.255	12.221	4.25	2.983
$B=3$	0.81	24.288	24.329	8.44	5.926

er than intrinsic uncertainties in the model which are much harder to estimate. Note that the energy ratios quoted in Table I are independent of f_π and ϵ. Once the Goldberger-Treiman relation has been used to establish the asymptotic form of $\theta(r)$, the Skyrme soliton is uniquely determined. In this light the energy of 1.4 GeV obtained for the $B=1$ Skyrme soliton seems a reasonable approximation to that combination of nucleon and $\Delta(1232)$ nucleon isobar which the hedgehog is presumed to describe in chiral bag models.

Following Skyrme, one can make a qualitative estimate of the interaction between two $B=1$ solitons at zero separation as

$$V_2(0) = E^{B=2} - 2E^{B=1}. \tag{7}$$

From Table I we see that $V_2(0)$ is essentially $E^{B=1}$ in remarkable agreement with Skyrme's early estimate. A similar estimate can be provided for the three-body interaction between three $B=1$ solitons at a point:

$$V_3(0,0) = E^{B=3} - 3E^{B=1} - 3V_2(0). \tag{8}$$

The results of Table I indicate that $V_3(0,0)$ is extremely small (on the order of $-0.025E^{B=1}$). Since chiral Lagrangians contain a variety of many-body forces,[10] it was not obvious that the present model would lead to the dominance of two-body forces familiar from low-energy nuclear physics. The results indicate that it does.

Our numerical solutions to Eq. (3) also permit determination of the baryon number density[1,2,6]

$$\rho_B(r) = -\frac{1}{2\pi^2}\frac{\sin^2\theta^B}{r^2}\frac{d\theta^B}{dr}. \tag{9}$$

Equation (9) is a perfect differential so that the baryon number, defined as the integral of ρ_B, is precisely B. Figure 1 shows $\rho_{B=1}(r)$ along with the related $\theta^{B=1}(r)$. Figure 1 also shows $\theta_{hh}(r)$ obtained from the (unstable) hedgehog solution with $\epsilon^2=0$ and the same asymptotic normalization. With use of Eq. (9) it is easy to calculate the rms baryon number radius. Results are shown in Table I. Again, specifying ϵ^2 fixes the rms radii uniquely so that the value of 0.48 fm obtained for $B=1$ does not seem unreasonable.

Finally, it is interesting to consider the energy of the $B=1$ state in a soliton bag model.[6] We do this rather arbitrarily by joining the present soliton field to a quark bag considered as a defect in the soliton field at $r=0.40$ fm where $\theta^{B=1}$ is precisely $\pi/2$. With this bag radius the quark kinetic energy is zero[7] and the energy of the soliton bag consists only of the energy of Eq. (2)

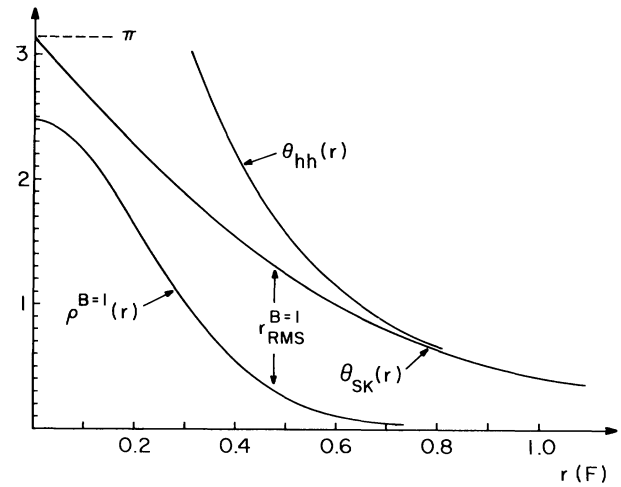

FIG. 1. The baryon density, $\rho^B(r)$, in arbitrary units and the chiral angle, $\theta^B(r)$, obtained for $\epsilon^2=0.005\,52$ and $B=1$. The chiral angle for the ordinary hedgehog, $\theta_{hh}(r)$, with the same asymptotic normalization is also shown.

from the region $\theta<\pi/2$ and the usual volume energy of the bag, $\mathcal{B}V_{bag}$. Picking $\mathcal{B}=1\ \text{fm}^{-4}$, the present solution yields an energy of 800 MeV for the $B=1$ state. (In this case precisely one-half of the baryon number is to be associated with the quarks; the other half is associated with the soliton.[6]) Indeed, the present soliton solutions can be used in any chiral bag calculation in place of the unstable solutions with $\epsilon^2=0$ previously employed. This instability, suggested by the factor of $|\epsilon|$ appearing in Eq. (5), forces the bag to play a dual role describing the physics of a quark phase and also curing a mathematical problem of the $\epsilon^2=0$ soliton in the limit of zero bag radius. It would seem reasonable to eliminate the latter role by employing the stable Skyrme soliton.

We have seen that setting ϵ^2 with the Goldberger-Treiman relation gives a unique prediction for the energy and size of the $B=1$ Skyrme soliton which is not unreasonable given the properties of nucleons and nucleon isobars. The ease with which these results were obtained suggests that Eq. (3) provides a formally stable and convenient description of the mesonic region in chiral bag models. The energy of the $B=2$ Skyrme soliton leads to a qualitatively sensible statement regarding the baryon-baryon interaction at zero separation and indicates a soft repulsion of about one baryon mass in magnitude.[11] This suggests that the old-fashioned wisdom[12] regarding the nucleon-nucleon interaction, as contained in chiral Lagrangians such as Eq. (1), may be adequate to describe all the qualitative features of the low-energy nucleon-nucleon interaction. We can imagine a $B=2$ solution in which the defects are separated by a large distance. In such a perturbative limit the usual one-pion-exchange interaction will appear. As the defects are moved closer, one expects to find the sigma meson of chiral models which provides the intermediate-range attraction common to all phenomenological models of the nucleon-nucleon interaction. At shorter distances, the nonlinear nature of the model dominates and, as we have seen, short-range repulsion results. The shortcomings in earlier models would appear to lie merely in their failure to provide an adequate nonlinear realization of these ideas at short distances. From the present results (not least the baryon density) it seems that the Skyrme soliton is describing much of the short-distance physics usually associated with a quark bag. Although we do not question the existence of a quark phase, it is not clear to what extent, if any, the limited needs of low-energy nuclear physics require its explicit inclusion.

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