PHYSICAL REVIEW LETTERS

VOLUME 51 29 AUGUST 1983 NUMBER 9

Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions

Horace P. Yuen

Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, Illinois 60201

(Received 27 June 1983)

The familiar minimum-uncertainty wave packets for masses are generalized in analogy with the two-photon coherent states of the radiation field. The free evolution of a subclass of these states, the contractive states, leads to a narrowing of the position uncertainty in contrast with the usual minimum-uncertainty wave packets. As a consequence the standard quantum limit for monitoring the positions of a free mass can be breached. Further implications on quantum nondemolition measurements are discussed.

PACS numbers: 03.65.Bz, 04.80.+z

There has been considerable recent interest in ascertaining and achieving the fundamental quantum limits on signal processing and precision measurements, in particular for applications to optical communications[1-3] and gravitational-wave detection.[4-8] A major result of this work is that one can beat the so-called standard quantum limit for amplitude measurements on harmonic oscillators. However, for the gravitational-wave interferometer[9] it is usually supposed[7,8] that the resolution is limited by the "standard quantum limit" (SQL) for measuring the positions of a free mass.[4-5] In this paper it is shown that the latter SQL is also *not* generally valid; it can be breached by a specific quantum measurement without special preparation of the free-mass quantum state. Toward this end I will describe a class of generalized minimum-uncertainty wave packets for masses, to be called twisted coherent states, which are also of interest in their own right. The breakdown of the SQL for free-mass position measurements demonstrates the fact that back actions from a conjugate observable do *not* necessarily, at least in accordance with the principle of quantum mechanics, limit the accuracy of subsequent measurements on an observable.

The evolution of a free mass is given by $X(t) = X(0) + P(0)t/m$, so that the position fluctuation at time t is

$$\langle \Delta X^2(t)\rangle = \langle \Delta X^2(0)\rangle + \langle \Delta P^2(0)\rangle t^2/m^2 + \langle \Delta X(0)\Delta P(0) + \Delta P(0)\Delta X(0)\rangle t/m. \quad (1)$$

In the previous derivation[4-5] of the general SQL for monitoring free-mass positions, it is implicitly assumed that the $t=0$ state of the mass (or the state after measurement) is such that the last term in (1) either vanishes or is positive. Under this assumption the uncertainty principle can be applied to minimize (1) at any time t with the resulting SQL

$$\langle \Delta X^2(t)\rangle_{\rm SQL} = \hbar t/m. \quad (2)$$

On the other hand, it is clear that $\langle \Delta X^2(t_0)\rangle = 0$ if the initial state is an eigenstate of the self-adjoint operator $X(0) + P(0)t_0/m$. Thus, the last term in (1) can surely be negative and the SQL is not generally valid. However, $\langle \Delta X^2(t)\rangle = 0$ implies $\langle \Delta P^2(t)\rangle = \infty$ so that $\langle P^2(0)/2m\rangle = \langle P^2(t)/2m\rangle = \infty$, i.e., an infinite average energy is needed to produce such a state.[3] A more realistic description can be developed as follows.

For an oscillator of mass m and frequency ω, the twisted or two-photon coherent states (TCS)[1,3]

$|\mu\nu\alpha\rangle$ are the eigenstates of $\mu a + \nu a^\dagger$:

$$(\mu a + \nu a^\dagger)|\mu\nu\alpha\rangle = (\mu\alpha + \nu\alpha^*)|\mu\nu\alpha\rangle,$$
$$|\mu|^2 - |\nu|^2 = 1, \tag{3}$$

where a is the annihilation operator of the oscillator mode. Here we adopt them to yield a class of states for a mass m with position X and momentum P. Define the following operator a on the Hilbert space of states for the mass:

$$a \equiv X(m\omega/2\hbar)^{1/2} + iP/(2\hbar m\omega)^{1/2}, \quad [a, a^\dagger] = I, \tag{4}$$

where ω is now an *arbitrary* parameter with unit $\sec^{-1}$. The *twisted coherent states* (TCS) $|\mu\nu\alpha\omega\rangle$ of a mass are defined to be the eigenstates of $\mu a + \nu a^\dagger$, $|\mu|^2 - |\nu|^2 = 1$, in analogy with (3) but with a given by (4). The free-mass Hamiltonian can be expressed

$$H = P^2/2m = \tfrac{1}{2}\hbar\omega(a^\dagger a - \tfrac{1}{2}a^2 - \tfrac{1}{2}a^{\dagger 2} + \tfrac{1}{2}). \tag{5}$$

The wave function $\langle x|\mu\nu\alpha\omega\rangle$, $X|x\rangle = x|x\rangle$, can be found through Eq. (3.24) of Ref. 1. Within the choice of a constant phase it is given by

$$\langle x|\mu\nu\alpha\omega\rangle = \left[\frac{m\omega}{\pi\hbar|\mu-\nu|^2}\right]^{1/4} \exp\left\{-\frac{m\omega}{2\hbar}\frac{1+i\xi}{|\mu-\nu|^2}\left[x-\left(\frac{2\hbar}{m\omega}\right)^{1/2}\alpha_1\right]^2 + i\left(\frac{2m\omega}{\hbar}\right)^{1/2}\alpha_2\left[x-\left(\frac{2\hbar}{m\omega}\right)^{1/2}\alpha_1\right]\right\}, \tag{6}$$

where

$$\xi \equiv \mathrm{Im}(\mu^*\nu); \quad \alpha \equiv \alpha_1 + i\alpha_2, \quad \alpha_1, \alpha_2 \text{ real}. \tag{7}$$

The wave functions (6) constitute a generalization of the usual minimum-uncertainty wave packets treated in every quantum mechanics textbook, which are given by (6) with $\xi = 0$. In the context of oscillators, "squeezing" is obtained when $\nu \neq 0$ in $|\mu\nu\alpha\rangle$, and ξ is related to the direction of minimum squeezing. As will be seen in the following, when $\xi > 0$ the x-dependent phase in (6) leads to a narrowing of $\langle\Delta X^2(t)\rangle$ from $\langle\Delta X^2(0)\rangle$ during free evolution, in direct contrast with the well-known spreading of $\langle\Delta X^2(t)\rangle$ for minimum-uncertainty wave packets.[10] Because of this behavior, mass states (6) with $\xi > 0$ will be called *contractive states*.

The first two moments of (6) are

$$\langle X\rangle \equiv \langle\mu\nu\alpha\omega|X|\mu\nu\alpha\omega\rangle = (2\hbar/m\omega)^{1/2}\alpha_1, \quad \langle P\rangle = (2\hbar m\omega)^{1/2}\alpha_2, \tag{8}$$

$$\langle\Delta X^2\rangle \equiv \langle(X - \langle X\rangle)^2\rangle = 2\hbar\zeta/m\omega, \quad \langle\Delta P^2\rangle = 2\hbar m\omega\eta, \tag{9}$$

$$\zeta \equiv |\mu - \nu|^2/4, \quad \eta \equiv |\mu + \nu|^2/4; \quad \zeta\eta = (1 + 4\xi^2)/16, \tag{10}$$

$$\langle\Delta X\Delta P\rangle = i\hbar/2 - \xi\hbar, \quad \langle\Delta P\Delta X\rangle = -i\hbar/2 - \xi\hbar, \tag{11}$$

$$\langle P^2/2m\rangle = \hbar\omega({\alpha_2}^2 + \eta). \tag{12}$$

The average mass energy (12) is finite when ω, α_2, and $|\nu|$ are finite. From (9)–(10) it follows that the minimum-uncertainty product $\langle\Delta X^2\rangle\langle\Delta P^2\rangle = \hbar^2/4$ is achieved if and only if $\xi = 0$.

The position fluctuation for a free mass starting in an arbitrary TCS (6) is immediately obtained from (1) and (9)–(11),

$$m\langle\Delta X^2(t)\rangle/2\hbar = \zeta/\omega - \xi t + \eta\omega t^2. \tag{13}$$

If $\xi \leq 0$, $\langle\Delta X^2(t)\rangle$ increases monotonically. In contrast to this usual situation, Eq. (13) is plotted in Fig. 1 for contractive states at $t = 0$ (i.e., for $\xi > 0$). The minimum fluctuation $1/16\omega\eta$ can be made arbitrarily small even for fixed ω by letting η (and thus also $\langle H\rangle$) become arbitrarily large. The time t_m at this fluctuation level is $t_m = \xi/2\eta\omega$ so that $m\langle\Delta X^2(t_m)\rangle/2\hbar t_m = 1/8\xi$. If $\langle\Delta X^2(t)\rangle$ is minimized with respect to ω at any given t simi-

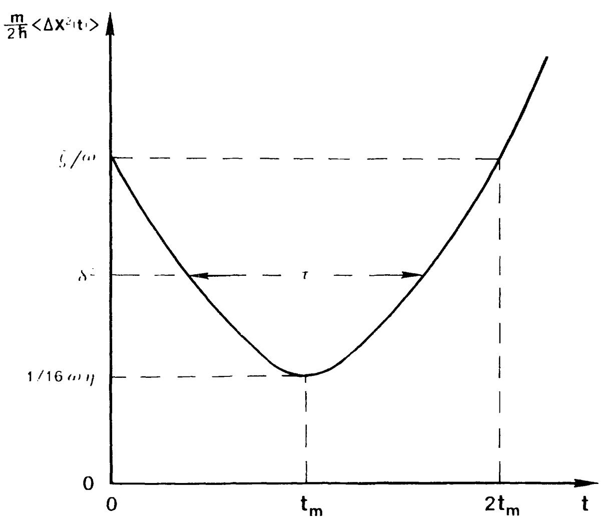

FIG. 1. The position fluctuation of a contractive state from (13); $t_m \equiv \zeta/2\eta\omega$, $t_m \to 0$ when $\xi \to 0$.

lar to the derivation of (2), one obtains

$$m\langle\Delta X^2(t)\rangle/2\hbar t=(\tfrac{1}{4}+\xi^2)^{1/2}-\xi \tag{14}$$

which, for large ξ, is approximately $1/8\xi$, the same value as that obtained from minimization with respect to t. The value $\langle\Delta X^2(t)\rangle/t$ given by (14) can be made arbitrarily small with a large ξ, in contrast with the SQL (2) or the case of ordinary minimum-uncertainty wave packets. Both $\xi\to\infty$ and $\eta\to\infty$ for obtaining small $\langle\Delta X^2(t)\rangle/t$ and $\langle\Delta X^2(t)\rangle$ can be achieved simultaneously by letting $|\nu|\to\infty$ with $\mathrm{Re}(\mu^*\nu)<0$.

The time interval τ for which $m\langle\Delta X^2(t)\rangle/2\hbar$ lies below a given level δ^2 satisfies, from (10) and (13),

$$\delta^2/\tau\geq\tfrac{1}{4}. \tag{15}$$

This constraint is relevant if one is interested in keeping the mass-position fluctuation as small as possible for as long as possible. Unlike the previous case (14), (15) is only a factor of 2 better than the SQL (2).

For application to sequences of position measurements, the $t=0$ free-mass state is that obtained immediately after a quantum measurement on the position of the mass. The measurement formalism of Gordon and Louisell[11] is used in the following discussion of quantum measurements. Thus, a quantum measurement is described by a set of operators $|\psi^S\rangle\langle\psi^M|$ such that $\langle\psi^M|\rho|\psi^M\rangle$ gives the measurement probability in state ρ while $|\psi^S\rangle$ is the state after measurement. The ordinary position measurement is then described by $|x\rangle\langle x|$, which is perfectly sharp and cannot be realized without an infinite average energy.[3] The measurement described by $|\mu\nu\alpha\omega\rangle\langle\mu\nu\alpha\omega|$ would have a position resolution $\hbar\xi/m\omega$ and simultaneously a momentum resolution $\hbar m\omega\eta$, while leaving the mass after measurement in $|\mu\nu\alpha\omega\rangle$. This measurement may be described as the measurement of a non-self-adjoint operator.[2-3,12-13] Nevertheless, it does *not* go beyond the framework of conventional quantum mechanics. Indeed, an explicit interaction Hamiltonian realization of this measurement in the standard fashion has been given before[14] for $\xi=0$, which can be generalized straightforwardly to arbitrary TCS's. Of course in this realization the initial apparatus and mass states are uncorrelated.

With this measurement one can monitor the positions of a free mass in time without suffering any back action from the mass momentum. One merely adjusts $\hbar\xi/m\omega$ to the required position resolution and makes the next measurement before a time lapse $\xi/\eta\omega$ (beyond which the position uncertainty of the next measurement would increase above the set level). This may be repeated indefinitely in a sequence of measurements all described by the same $|\mu\nu\alpha\omega\rangle\langle\mu\nu\alpha\omega|$, during which the position uncertainty never increases beyond the set level. There is no need to intervene between measurements for state preparation. However, there is a limitation $\hbar/m$ on the ratio of the resolution $\hbar\xi/m\omega$ and the time lapse $\xi/\eta\omega$ from (15). Even this limitation is overcome in a measurement described by $\langle\mu\nu\alpha\omega| \times \langle\mu'\nu'\alpha\omega'|$, in which different values of μ', ν', and ω' are used to set the required position resolution and μ, ν, ω are used to adjust the time lapse. While such a measurement is possible in principle,[11] no explicit Hamiltonian realization is known.

The breakdown of the general SQL does not imply that the particular gravitational-wave interferometer is not subject to a serious resolution limit of the order given by (2). In fact Caves[7,8] has produced a separate argument for the validity of (2) from a specific analysis of the interferometer. On the other hand, a recent analysis[15] indicates that there is no limit to the resolution of such interferometers. It appears that a careful and complete quantum mechanical treatment of the interferometer is in order.

The above analysis shows that back actions from a conjugate observable need not induce any inaccuracy in the subsequent measurement of an observable. Even though the position measurement of a free mass is not a quantum nondemolition (QND) measurement and the position operator not a QND observable according to the recent refined definitions,[4,6] it is clear from the above development that they may still have the QND character in the original sense of Braginskiĭ and Vorontsov[5]—namely, the disturbance due to the first measurement plus subsequent free motion do not demolish the possibility of an accurate second measurement. In this primal and useful sense no one has ever shown, for either a free mass or an oscillator, that there is any observable which cannot be monitored in a QND way. Indeed, the above analysis can be construed as a way to perform QND position measurements on a free mass.

I would like to thank R. S. Bondurant, J. H. Shapiro, and R. Weiss for stimulating discussions. This work was supported by the U. S.

Office of Naval Research.

[1]H. P. Yuen, Phys. Rev. A 13, 2226 (1976).

[2]H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory 26, 78 (1980).

[3]H. P. Yuen, in *Quantum Optics, Experimental Gravitation, and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983).

[4]C. M. Caves *et al.*, Rev. Mod. Phys. 52, 341 (1980).

[5]V. B. Braginskiĭ and Yu. I. Vorontsov, Usp. Fiz. Nauk 114, 41 (1974) [Sov. Phys. Usp. 17, 644 (1975)].

[6]W. G. Unruh, Phys. Rev. D 19, 2888 (1979).

[7]C. M. Caves, Phys. Rev. D 23, 1693 (1981).

[8]C. M. Caves, in *Quantum Optics, Experimental Gravitation, and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983).

[9]See, e.g., R. Weiss, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge, Univ. Press, Cambridge, 1979).

[10]See, e.g., L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), 3rd ed., pp. 60–64.

[11]J. P. Gordon and W. H. Louisell, in *Physics of Quantum Electronics*, edited by P. L. Kelley *et al.* (McGraw-Hill, New York, 1966), pp. 833–840.

[12]H. P. Yuen and M. Lax, IEEE Trans. Inf. Theory 19, 740 (1973).

[13]H. P. Yuen, Phys. Lett. 91A, 101 (1982).

[14]E. Arthurs and J. L. Kelly, Jr., Bell Syst. Tech. J. 44, 725 (1965).

[15]R. S. Bondurant and J. H. Shapiro, in *Coherence and Quantum Optics V*, edited by L. Mandel and E. Wolf (Plenum, New York, 1983).