Invasion Percolation on the Cayley Tree: Exact Solution of a Modified Percolation Model

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A modified form of percolation theory is solved exactly on a Cayley tree of arbitrary coordination number. In this model, cluster growth proceeds "dynamically" by invasion along a path of least resistance. Although there are clear parallels to ordinary "static" percolation at the percolation threshold, there are also significant differences. In particular, the scaling function which describes the shape of large clusters is different in the two models.

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In this Letter we consider a modified form of percolation theory known as invasion percolation. This model was motivated by the study of the displacement of one fluid by another in a porous medium,[1] but in principle it may be applied to any kind of "dynamical" percolation which proceeds by invasion along a path of least resistance. A detailed description of the model has recently been given by Wilkinson and Willemsen[2] (hereafter WW). The major focus of these papers was the process of invasion in a finite domain, with a trapping rule which prevents the displacing fluid from invading a region which it has surrounded, this representing physically the incompressibility of the displaced phase. Here we will consider the somewhat simpler problem of growing a cluster from a point into an infinite medium in the absence of this trapping rule. For a lattice representation of the medium, this process may be defined as follows: (1) Consider an infinite lattice of sites and connecting bonds in which every site is assigned a random number r, drawn from a uniform distribution on the unit interval $0 \leq r < 1$. (2) Choose a site (the "origin") to be occupied as the start of the cluster. Define the boundary to be those unoccupied sites which are nearest neighbors of sites in the cluster. (3) At each time step increase the number of sites in the cluster by one, by occupying that boundary site which has the smallest random number.

These rules may be contrasted with the operational definition of ordinary percolation in which the decision whether a given site is occupied or not is also made by examining a random number. From this viewpoint the algorithm for growing ordinary percolation clusters at occupation probability p is to take rules 1 and 2 above but replace rule 3 by the following: (3′) At each time step the current boundary sites are examined and those with random number r less than p are accepted into the cluster. The cluster terminates when no sites with r less than p remain on the boundary. As is well known[3] there is a critical percolation probability p_c such that for $p<p_c$ the cluster always terminates, but for $p>p_c$ there is a finite probability that the cluster grows indefinitely.

There is no control variable in invasion percolation analogous to p, but that the two forms of percolation are not so different may be seen by considering the invasion-percolation acceptance probability profile $a_n(r)$, defined[4] such that $a_n(r)\,dr$ is the probability that the random number chosen at the nth step is in the interval $[r, r+dr]$. It was found by WW that as $n\to\infty$, the profile approaches a step function of the form

$$a_\infty(r)=\begin{cases} p_c^{-1} \text{ if } r<p_c, \\ 0 \text{ if } r>p_c, \end{cases} \tag{1}$$

where p_c is the ordinary percolation threshold for the lattice in question. This apparently remarkable result[5] is not really so surprising, as one knows from ordinary percolation that the invasion cluster could not grow to $n=\infty$ if only random numbers less than p_c were picked. One can also argue heuristically that as $n\to\infty$ the law of large numbers precludes the necessity of picking a finite fraction of random numbers larger than p_c.

The central result (1) suggests that large invasion-percolation clusters are "the same" as ordinary percolation clusters when p is exactly p_c, since the distribution of random numbers accepted into the cluster is the same in the two cases. However, this is a somewhat vague statement, and the main purpose of the present Letter is to examine its meaning in an exactly soluble case.

What follows is an exact analysis of invasion percolation on the Cayley tree, and a comparison of these results with those of ordinary percolation. We define a tree of coordination number $1+\sigma$ to consist of σ^m sites (nodes) on level m in which every node on level $m\neq 0$ is considered a nearest neighbor of one node on level $m-1$ and

σ nodes on level $m+1$. The origin is the root node at $m=0$, and the first step, $n=1$, consists of adding to the cluster one of the σ nodes at level $m=1$. Since the analysis is of necessity somewhat technical, we will summarize the main results here:

(a) The result (1) is correct with $p_c=\sigma^{-1}$.

(b) The probability that a random number greater than $\sigma^{-1}+\epsilon$ is chosen on the nth step vanishes as $n\to\infty$ as $1/\sqrt{n}$ for $\epsilon=0$ and exponentially in n for $\epsilon>0$.

(c) Define the shape function $S_m{}^n$ for a cluster of size n to be the number of occupied sites on level m of the tree. Then as $m, n\to\infty$ with $m/\sqrt{n}$ fixed, $S_m{}^n$ has the scaling form

$$S_m{}^n=\left(\frac{2(\sigma-1)n}{\sigma}\right)^{1/2}\hat{S}\left(m\left(\frac{\sigma-1}{2\sigma n}\right)^{1/2}\right), \qquad (2)$$

which is of the same form as in ordinary percolation.

(d) The scaling function $\hat{S}(x)$ is independent of σ but is different from that of ordinary percolation.

Our method of solution relies heavily on the use of generating functions, whose application to percolation on the Cayley tree is well known.[6] However, to our knowledge, detailed questions of cluster shapes have not been investigated before, and so we begin with a derivation of the results in ordinary percolation which we need for later comparison. If a cluster of n nodes is connected to the root, it will necessarily be surrounded by exactly $(\sigma-1)n+\sigma$ vacant boundary nodes. This implies that every such connected cluster occurs with equal probability $p^n(1-p)^{(\sigma-1)n+\sigma}$ and enables us to determine average cluster properties at fixed n without reference to the value of p. To compute $S_m{}^n$ we introduce the generating function

$$f(\beta)=1+\beta+\sigma\beta^2+\ldots=1+\beta[f(\beta)]^\sigma,$$

in which the coefficient of β^n counts the number of possible n-node clusters connected to the root through *one* of the σ branches. The generating function for the total number of clusters connected to the root through all σ branches is $[f(\beta)]^\sigma$, and for the number of clusters containing a particular node on level m is $\beta^m[f(\beta)]^{(\sigma-1)m+\sigma}$. The probability that a particular node on level m is one of n nodes connected to the root is the ratio of the coefficients of β^n in these two expressions, and these are easily determined by contour integration using f as the independent variable.[6] Multiplying this probability by σ^m we obtain the expected number

$$S_m{}^n=\sigma^{m-1}(\sigma m-m+\sigma)\,\frac{n!\,(n\sigma-m+\sigma-1)!}{(n-m)!\,(n\sigma+\sigma-1)!}. \qquad (3)$$

The mean level number $\langle m\rangle=n^{-1}\sum_m mS_m{}^n$, which is the analog of the mean square cluster size R^2 on a regular lattice, can be expressed as the single hypergeometric function polynomial $\langle m\rangle=F(1, 1-n; 1-\sigma-n\sigma;\sigma)$ and asymptotically

$$\langle m\rangle=\left(\frac{n\pi\sigma}{2(\sigma-1)}\right)^{1/2}-\frac{4\sigma+1}{3(\sigma-1)}+o(1). \qquad (4)$$

Finally, in the limit $n\to\infty$, $m/\sqrt{n}$ fixed, the shape function (3) reduces to the scaling form (2) with

$$\hat{S}(x)=xe^{-x^2}. \qquad (5)$$

Thus any measure of the "linear" extent of these large clusters must scale as $R\sim(\langle m\rangle)^{1/2}\sim n^{1/\varphi}$, with the "classical" fractal dimension $\varphi=4$.

To obtain the corresponding shape function for invasion percolation we define the generating functions

$$g_m(\beta;r)=\sum_{n=0}^{\infty}\beta^n g_m{}^n(r),$$

$$g(\alpha,\beta;r)=\sum_{m=0}^{\infty}\sigma^m\alpha^m g_m(\beta;r),$$

where $g_m{}^n(r)dr$ is the probability that a particular node on level m has an associated random number in the interval $[r, r+dr]$ and is selected to be added to the growing cluster at step n. We define $g_m{}^0(r)=\delta_{m0}$. The integrated quantity $G_m{}^n=\int_0^1 dr\, g_m{}^n(r)$ is the probability that a given node on level m is added at step n, and the shape function $S_m{}^n=\sigma^m\sum_{n'=m}^n G_m{}^{n'}$. Consider first the case $m=1$. For an $m=1$ node on one branch to be selected at the nth step, the root must be connected to an $n-1$ cluster through the remaining $\sigma-1$ branches. Furthermore, that cluster must be an ordinary percolation cluster in that every cluster node has random number less than r, and every boundary node has random number greater than r, so that the probability density $g_1{}^n(r)$ will contain the factor $r^{n-1}(1-r)^{(\sigma-1)n}$. Knowing that the number of these ordinary percolation clusters is described by the generating function $[f(\beta)]^{\sigma-1}$ we conclude that $g_1(\beta;r)=\beta(1-r)^{\sigma-1}[f(\beta r(1-r)^{\sigma-1})]^{\sigma-1}$. For $\sigma=2$ this gives $g_1(\beta;r)=\{1-[1-4\beta r(1-r)]^{1/2}\}/2r$. The generating function for $m>1$ can be obtained by induction. Assume that we have determined $g_{m-1}(\beta;s)$ for a node with random number s on level $m-1$, and wish to determine the probability that one of its neighbors on m has random number in the interval $[r, r+dr]$ and is to be selected. The other $\sigma-1$ nearest neighbors on level m must be part of an ordinary percolation cluster, so that $g_m(\beta;r)$ will contain a factor $g_1(\beta;r)$. To complete the calculation we must distinguish between the cases $s>r$ and $s<r$. In the former, no additional growth in the branches described by $g_{m-1}(\beta;s)$ is possible, while in the latter such additions are simply handled by making the replacement $g_{m-1}(\beta;s)\to g_{m-1}(\beta;r)$. In this way we obtain the recursion relation and corresponding integral equation:

$$g_m(\beta;r)=g_1(\beta;r)[\int_0^r ds\, g_{m-1}(\beta;r)+\int_r^1 ds\, g_{m-1}(\beta;s)], \tag{6a}$$

$$g(\alpha,\beta;r)=1+\sigma\alpha g_1(\beta;r)[rg(\alpha,\beta;r)+\int_r^1 ds\, g(\alpha,\beta;s)]. \tag{6b}$$

The integral Eq. (6b) can be solved explicitly to yield

$$g(\alpha,\beta;r)=1+\frac{\sigma\alpha g_1(\beta;r)}{1-\sigma\alpha r g_1(\beta;r)}\exp\int_r^1\frac{\sigma\alpha g_1(\beta;s)}{1-\sigma\alpha s g_1(\beta;s)}ds. \tag{7}$$

This is the fundamental result of this Letter. Although the integral in (7) can be expressed in terms of elementary functions for the special case $\sigma=2$, it suffices for certain scaling properties to evaluate it for α and β near unity only.

The distribution of random numbers selected at step n is $a_n(r)=\sum_m\sigma^m g_m{}^n$, which is the coefficient of β^n in $g(1,\beta;r)$. For $\sigma=2$ we find

$$g(1,\beta;r)=\{1+\beta(1-2r)/[1-4\beta r(1-r)]^{1/2}\}/(1-\beta),$$

while for general σ the dominant singularities in the limit $\beta\to1$ are similar, and lead to the large-n scaling behavior of the acceptance profile as

$$a_n(r)\sim\tfrac{1}{2}\sigma\,\mathrm{erfc}\{(\sigma r-1)[n\sigma/2(\sigma-1)]^{1/2}\}. \tag{8}$$

As postulated by WW, this profile depends only on the scaling variable $(r-p_c)n^{1/\Delta}$ with the classical gap exponent $\Delta=2$. Equation (8) is the basis for the statements (a) and (b) in the introduction.

It follows from (6b) that the integrated quantity

$$G(\alpha,\beta)=\int_0^1 dr\, g(\alpha,\beta;r)=\sum_{m,n}\sigma^m\alpha^m\beta^n G_m{}^n$$

is given by $[g(\alpha,\beta;0)-1]/\sigma\alpha g_1(\beta;0)$, i.e., no additional integration beyond that in (7) is required. For $\sigma=2$ we find

$$G(\alpha,\beta)=\exp\left\{\frac{\alpha}{1-2\alpha}\left[\ln(1-\beta)+\frac{\beta(1-\alpha)}{Q}\ln\left(\frac{1-\alpha\beta+Q}{1-\alpha\beta-Q}\right)\right]\right\}, \tag{9}$$

where $Q=[\beta(1-2\alpha+\alpha^2\beta)]^{1/2}$. For arbitrary σ, the integrand in (7) can be expanded in a Taylor series with the necessary coefficients easily determined by the same methods that led to (3). The integration over s is then trivial and we find

$$G(\alpha,\beta)=\exp\left\{(\sigma-1)\sum_{m,n} m\sigma^m\frac{(n-1)!(n\sigma-m-1)!}{(n-m)!(n\sigma)!}\alpha^m\beta^n\right\}. \tag{10}$$

From (10) we obtain the mean level number

$$\langle m\rangle=\frac{1}{n}\sum_{n'=1}^{n}\sum_{n''=1}^{n'}\frac{1}{n''}F(1,1-n'';1-n''\sigma;\sigma)=\frac{4}{3}\left[\frac{n\pi\sigma}{2(\sigma-1)}\right]^{1/2}-\frac{1}{3}\frac{\sigma+1}{\sigma-1}\ln n+O(1), \tag{11}$$

which to leading order as $n\to\infty$ is larger by a factor $\frac{4}{3}$ than that for ordinary percolation as given by (4). Finally, the behavior of $G_m{}^n$ for large m and n is determined by the singularity structure of $G(\alpha,\beta)$ as $\alpha\to1$ and $\beta\to1$. With $t^2=\sigma(1-\alpha)^2/2(1-\beta)(\sigma-1)$ fixed, we find

$$G(\alpha,\beta)\sim\frac{1}{1-\beta}\exp\left\{\frac{-2t}{(t^2-1)^{1/2}}\ln[t+(t^2-1)^{1/2}]\right\}, \tag{12}$$

a result which agrees with (9) when $\sigma=2$. The scaling form (12) implies the scaling form (2) for $S_m{}^n$

with

$$\hat{S}(x) = 2x\int_x^\infty dy\, \hat{G}(y)/y^2, \tag{13a}$$

$$\hat{G}(x) = \frac{1}{2\pi i}\,\frac{1}{\sqrt{\pi}}\int_C dt \exp\left\{-x^2t^2 - \frac{2t}{(t^2-1)^{1/2}}\ln[t+(t^2-1)^{1/2}]\right\}, \tag{13b}$$

where the contour encloses, counterclockwise, the cut in the integrand, $-\infty < t \leq -1$. Clearly no rescaling can make the invasion percolation expression (13a) equal to the simple Gaussian (5) for ordinary percolation. For small x, $\hat{S}(x)$ in Eq. (13a) has the behavior $x/2 + O(x^3)$, which translates into the statement that for $1 \ll m \ll \sqrt{n}$ only *one half* as many nodes are occupied by invasion as compared to ordinary percolation. For general x, see Fig. 1. The differences between the two models are by no means small, and we indeed consider this result the *paradox of invasion percolation* in view of the sharp-profile result (8).

Since the above analysis has been somewhat involved, we have thought it worthwhile to check the results by direct Monte Carlo simulation. In particular, based on 10 000 realizations, we find the cluster shape function $S_m{}^n$ for $1 \leq m \leq n \leq 100$ for $\sigma = 2$ and $\sigma = 3$ to be in good agreement with the analytic results obtained from (10). For regular lattices, Monte Carlo simulation seems the only way to obtain quantitative information about invasion percolation, since an exact analysis along the lines of this Letter is clearly impossible, and even series-expansion methods appear difficult to apply. One interesting aspect of such simulations is that, because of the fundamental property (1), they can provide an extremely good estimate of p_c. Preliminary results for the square- and simple-cubic lattices yield values which are competitive with best known estimates from series and Monte Carlo work on ordinary percolation. Two additional quantities which can be obtained from these simulations are the root mean square cluster radius R, and the "tail" of the acceptance profile, $A = \int_{p_c}^1 dr\, a(r)$. If we assume that as the cluster size $n \to \infty$ these scale as $R \sim n^{1/\varphi}$ and $A \sim n^{-\lambda}$, then the exponents φ and λ can be estimated. We conjecture[7] that the fractal dimension φ is the same as that of ordinary percolation, i.e., Δ/ν, where Δ and ν are the usual gap and correlation length exponents, and that λ is $1/\Delta$. These conjectures are consistent with our results on the Cayley tree, for which $\Delta = 2$ and $\nu = \frac{1}{2}$, and also with preliminary Monte Carlo simulations in two and three dimensions. Further Monte Carlo work is in progress.

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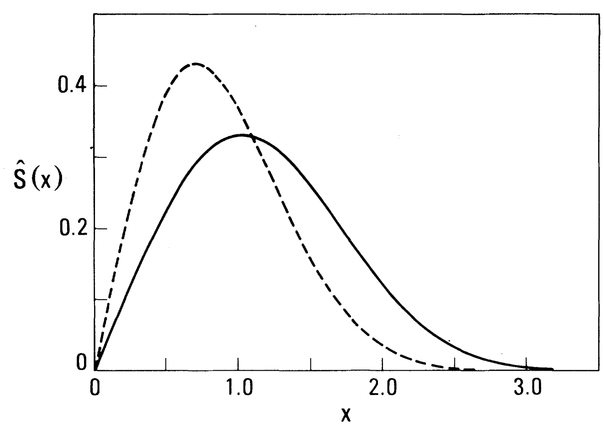

FIG. 1. Cluster-shape scaling curves. The dashed line is ordinary percolation, Eq. (5), and the solid line invasion precolation, Eq. (13a).

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[2]D. Wilkinson and J. Willemsen, to be published.

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[4]For technical reasons WW used a slightly different definition.

[5]Recently we have been shown an outline of a rigorous proof of (1) by C. Newman.

[6]M. E. Fisher and J. W. Essam, J. Math. Phys. (N.Y.) 2, 609 (1961).

[7]Analogous conjectures for invasion percolation on a finite lattice were made by WW, Ref. 2.