Method of Solution for a Class of Multidimensional Nonlinear Evolution Equations

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A general method is given for solving certain inverse problems in the plane. The results can be used to construct the solution to the initial-value problems of related nonlinear evolution equations in two spatial and one temporal dimension. The method also allows one to compute lumps, i.e., multidimensional solitons tending to zero in all spatial directions.

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The use of inverse scattering in one dimension for solving the initial-value problem of certain nonlinear equations in $1+1$ dimensions, i.e., one spatial and one temporal dimension, has been well established.[1] A central idea is to relate a nonlinear equation to a pair of linear operators, the so-called Lax pair.[2] One of these operators is "time independent" and is considered as a scattering (or eigenvalue) problem. Well-known scattering problems are the Schrödinger scattering problem,[3] the so-called Ablowitz-Kaup-Newell-Segur (AKNS)[4] system (a system of two equations), and their natural generalizations: the $n\times n$ AKNS[5] and the Gel'fand-Dikii[6] operator. The inverse problem of the last two has only recently been solved.[7] Several physically important equations, e.g., the Korteweg-de Vries, sine-Gordon, nonlinear Schrödinger, Boussinesq, n-wave interaction, etc., can be linearized via the above scattering problems.

It is also well known[1] that certain two-dimensional generalizations of the above scattering equations are related to physically interesting nonlinear equations in $2+1$ dimensions [these equations are, in a sense, the $(2+1)$-dimension analogs of the nonlinear equations mentioned above]. In particular the Schrödinger scattering problem has been generalized and it is then related to the Kadomtsev-Petviashvili (KP) equation; there are two important cases, the so-called KP I and KP II, which differ by a crucial sign. Similarly, the generalized $n\times n$ AKNS system[5] can be either hyperbolic or elliptic. In both cases it is related to several physically significant multidimensional nonlinear equations.

In spite of this connection between nonlinear equations in $2+1$ dimensions and linear systems in the plane, the question of finding a viable method, such as that in $1+1$ dimensions, i.e., the inverse scattering transform (IST), for solving the initial-value problem of these and other multidimensional equations has essentially remained open. It should be noted that interesting results in this direction were given in Refs. 8 and 9.

In this Letter we outline a rather general method for solving certain important inverse problems in the plane as well as the initial-value problem of the corresponding nonlinear equation. This method has emerged from our investigation of the scattering problems associated with several concrete problems: Benjamin-Ono,[10] KP I,[11] KP II,[12] hyperbolic, and elliptic systems.[13,14] We point out that the treatment of KP II was of crucial importance for the development of this method, since it was the first time that the inadequacy of the Riemann-Hilbert formulation of the IST in $2+1$ dimensions was discovered. In this Letter we give the linear integral equations associated with the solution of the inverse problem of the following scattering equations: the generalized Schrödinger equations related to KP I and KP II; the hyperbolic and elliptic versions of the generalized $n\times n$ AKNS system, which are related to the Davey-Stewartson (DS) equation[15] [a $(2+1)$-dimension generalization of the nonlinear Schrödinger equation in $1+1$ dimensions], the n-wave interaction in $2+1$ dimensions, the modified KP equation, etc. Details can be found in Refs. 11–14.

The main steps of the IST for problems in $2+1$ dimensions can be summarized as follows: (i) Define an eigenfunction $\mu(x,y,k)$ which is bounded for all complex values of the "spectral parameter" k and which is appropriately normalized ($\mu\to I$ as $k\to\infty$). This eigenfunction is usually defined in terms of a Fredholm linear integral equation; it may have homogeneous solutions. (ii) Compute $\partial\mu/\partial\bar{k}$. This is, in general, expressed in terms of some other bounded eigenfunction, which we call $N(x,y,l,k)$, and appropriate scattering data. We note that in some problems (e.g., Benjamin-Ono, KP I) $\mu(x,y,k)$ is a *sectionally meromorphic function of* k, i.e., it is holomorphic modulo poles, in regions of the complex k plane separated by certain contours, and it has a jump across these contours. In these cases $\partial\mu/\partial\bar{k}$, which measures the "departure of $\mu(x,y,k)$ from holomorphicity," will be zero everywhere except on the pole locations and on the above contours. (iii) Employ a suitable "symmetry" relationship between N and μ to express $\partial\mu/\partial\bar{k}$ in terms of μ and appropriate data which by analogy to one dimension we call scattering data. If μ has homogeneous solutions then one needs also to establish a relationship between μ and these homogeneous modes. We have so far encountered two types of symmetry conditions: "discrete" (KP II, elliptic systems) and "differential" (Benjamin-Ono, KP I). The relationship between $\partial\mu/\partial\bar{k}$, μ, and the scattering data is the central equation associated with the inverse problem of a given equation. It defines, in general, a "$\bar{\partial}$" problem, i.e., given $\partial\mu/\partial\bar{k}$ find μ. In the case that μ is sectionally meromorphic this "$\bar{\partial}$" problem degenerates to a Riemann-Hilbert problem. (iv) Use the following extension of Cauchy's formula[16]

$$\mu(x,y,k)=\frac{1}{2\pi i}\iint_R\frac{[\partial\mu(x,y,z)/\partial\bar{z}]dz\wedge d\bar{z}}{z-k}+\frac{1}{2\pi i}\int_C\frac{\mu(x,y,z)}{z-k}dz \tag{1}$$

(where R and C are an appropriate region and contour in the z plane, respectively) to solve the "$\bar{\partial}$" problem. Its solution is found, in general, in terms of a linear integral equation for $\mu(x,y,k)$. Equation (1) is uniquely defined in terms of the above-mentioned scattering data. (v) Calculate the potential $q(x,y)$ directly from the solution of the inverse problem [typically given by integrals over $\mu(x,y,k)$ and the scattering data]. The above discussion summarizes the steps needed for the solution of the inverse problem. (vi) To effect the solution of the related nonlinear evolution equation one needs to employ the other linear operator in the Lax pair, i.e., the time-dependent part. The action of this operator fixes the time evolution of the scattering data in terms of initial scattering data. Since the initial scattering data can always be expressed in terms of the initial data $q(x,y,0)$, Eq. (1), and hence the formula for $q(x,y,t)$, is uniquely defined in terms of the initial data.

(a) We first consider the scattering equation

$$\sigma\mu_y+\mu_{xx}+2ik\mu_x+q(x,y)\mu=0, \tag{2}$$

where $\sigma=i$, and $\mu(x,y,k)$ and $q(x,y)$ are scalars. We assume that $q(x,y)\to 0$ sufficiently fast as $x^2+y^2\to\infty$. The solution of the inverse problem associated with (2) is given by the following linear integral equations:

$$\mu^-(k)-i\sum_{l=1}^{N}\left(\frac{\varphi_l^+}{k-k_l^+}+\frac{\varphi_l^-}{k-k_l^-}\right)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\nu,l)e^{\theta(x,y,l,\nu)}\mu^-(l)\,dl\,d\nu}{\nu-k+i0}=1, \tag{3a}$$

$$(x-2k_j^{\pm}y+\gamma_j^{\pm})\varphi_j^{\pm}-i\sum_{l=1}^{N}{}'\left(\frac{\varphi_l^+}{k_j^{\pm}-k_l^+}+\frac{\varphi_l^-}{k_j^{\pm}-k_l^-}\right)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\nu,l)e^{\theta(x,y,l,\nu)}\mu^-(l)\,dl\,d\nu}{\nu-k_j^{\pm}}=1, \tag{3b}$$

where $\mu^-(k)\equiv\mu^-(x,y,k)$, $\varphi_l^{\pm}=\varphi_l^{\pm}(x,y)$, $\theta(x,y,l,k)\equiv i(l-k)x-i(l^2-k^2)y$, and $\sum'$ means summation from $\sigma=1$ to N unless any of the denominators vanish (i.e., omit those terms). Equations (3) define $\mu^-(x,y,k)$, $\{\varphi_j^+(x,y),\ \varphi_j^-(x,y)\}_{j=1}^N$ in terms of $f(k,l)$, $\{k_j^{\pm},\gamma_j^{\pm}\}_{j=1}^N$, which can be expressed in terms of integrals over suitable scattering functions. Once μ^- and $\varphi_j^{\pm}$ have been found, $q(x,y)$ is reconstructed from

$$q(x,y)=\frac{\partial}{\partial x}\left\{2\sum_{1}^{N}[\varphi_j^+(x,y)+\varphi_j^-(x,y)]+\pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(k,l)e^{\theta(x,y,l,k)}\mu^-(x,y,l)\,dk\,dl\right\}. \tag{4}$$

Let the scattering data evolve in time as follows: $\partial k_j^{\pm}/\partial t=0$, $\gamma_j^{\pm}(t)=12(k_j^{\pm})^2t+\gamma_j^{\pm}(0)$, $f(k,l,t)=f(k,l,0)\exp[4i(l^3-k^3)t]$. Then $q(x,y,t)$ as defined from Eq. (4) [where μ^- and $\varphi_j^{\pm}$ are computed via (3)] solves the KP I equation, i.e.,

$$(q_t+6qq_x+q_{xxx})_x=-3\sigma^2q_{yy}, \tag{5}$$

with $\sigma=i$. Pure lump solutions correspond to $f(k,l,0)=0$, in which case (3) reduces to a system of algebraic equations; hence the potential q is found in closed form.

(b) The solution of the inverse problem associated with (2) with $\sigma=-1$ is given by

$$\mu(x,y,k)=1+\frac{1}{2\pi i}\iint_{R_\infty}\frac{F(z_R,z_I)\exp[-i(2z_Rx-4z_Iz_Ry)]\,\mu(x,y,-\bar{z})\,dz\wedge d\bar{z}}{z-k}, \tag{6}$$

where R_∞ is the entire z-complex plane, $\bar{z}=z_R-iz_I$, $dz\wedge d\bar{z}=-2i\,dz_R\,dz_I$. Equation (5) defines μ in terms of $F(z_R, z_I)$ which as above is obtained in closed form, i.e.,

$$F(k_R, k_I) = -\operatorname{sgn}(k_R)\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \exp[i(2k_R\xi - 4k_I k_R \eta)] q(\xi,\eta)\mu(\xi,\eta,k)/(4\pi).$$

Then $q(x,y)$ is found from

$$q(x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\iint_{R_\infty} F(z_R, z_I)\exp[-i(2z_R x - 4z_I z_R y)]\,\mu(x,y,-\bar{z})\,dz\wedge d\bar{z}. \tag{7}$$

Let $F(z_R, z_I, t) = F(z_R, z_I, 0)\exp[-4i(z^3 + \bar{z}^3)t]$. Then $q(x,y,t)$ as defined from Eq. (7), solves KP II, i.e., Eq. (5) with $\sigma = -1$.

(c) Consider the hyperbolic version of the system

$$\mu_x = ik\hat{J}\mu + q\mu + \sigma J\mu_y, \quad \hat{J}f \equiv Jf - fJ, \tag{8}$$

i.e., $\sigma = 1$. In Eq. (8) $\mu(x,y,k)$ is an nth-order matrix, J is a constant real diagonal matrix with elements $J_1 > J_2 > \cdots > J_n$, and $q(x,y)$ is an nth-order off-diagonal matrix containing the potentials $q_{ij}(x,y)$. We assume that $q_{ij}(x,y) \to 0$ sufficiently fast as $x^2 + y^2 \to \infty$. The inverse problem associated with Eq. (8) can be solved via

$$\mu^-(x,y,k) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mu^-(x,y,l)e^{ilJx}f(l,\nu)e^{-i\nu Jx + i(l-\nu)y}\,dl\,d\nu}{\nu - k + i0} = I, \tag{9}$$

where I is the $n\times n$ identity matrix and as above $f(l,\nu)$ is obtained in closed form. The potential is reconstructed from

$$q(x,y) = -(1/2\pi)\hat{J}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^-(x,y,l)e^{ilJx}f(l,\nu)e^{-i\nu Jx + i(l-\nu)y}\,dl\,d\nu. \tag{10}$$

Let $f(l,k,t) = \exp(iltA)f(l,k,0)\exp(-iktA)$, where $A = \operatorname{diag}(C_1,\ldots,C_n)$ and J_i, C_i are defined in terms of α_{ij}, β_{ij} via $\alpha_{ij} = (C_i - C_j)/(J_i - J_j)$, $\beta_{ij} = C_i - J_i\alpha_{ij}$. Then $q(x,y,t)$ as defined from (10) solves the n-wave interaction equations in 2+1 dimensions,

$$q_{ij_t} = \alpha_{ij}q_{ij_x} + \beta_{ij}q_{ij_y} + \sum_{k=1,\,k\neq j}^{n}(\alpha_{ik} - \alpha_{kj})q_{ik}q_{kj}, \quad i,j,k = 1,\ldots,n. \tag{11}$$

Let $J = \operatorname{diag}(1,-1)$, $q_{12} = Q$, $q_{21} = \sigma\bar{Q}$, $\sigma = \pm 1$, $f(l,k,t) = \exp(-l^2 tA)f(l,k,0)\exp(k^2 tA)$, $A = \operatorname{diag}(i,-i)$. Then Q satisfies the DS I equation,

$$iQ_t + \tfrac{1}{2}(Q_{xx} + Q_{yy}) = -\sigma|Q|^2 Q + \varphi Q; \quad \varphi_{xx} - \varphi_{yy} = 2\sigma(|Q|^2)_{xx}, \quad \sigma = \pm 1. \tag{12}$$

(d) Consider the elliptic version of (8), i.e., $\sigma = -i$. The solution of its inverse problem is given by the following linear integral equations:

$$\mu(x,y,k) - [T_{x,y,\Omega}\,\mu(x,y,\cdot)](k) - S(x,y,k) = I, \tag{13a}$$

$$(-ixJ_i + y + \gamma_{l_i}{}^i)\underline{\varphi}_{l_i}{}^i - \underline{\hat{T}}_{l_i}{}^i(x,y) - \underline{\hat{S}}_{l_i}{}^i(x,y) = \underline{I}^i, \quad l_i = 1,2,\ldots,\Lambda_i, \tag{13b}$$

for $i = 1,\ldots,n$. In Eq. (13a) the matrix operator $T_{x,y,\Omega}$ is defined by

$$[T_{x,y,\Omega}\,g(\cdot)](k) \equiv \frac{1}{2\pi i}\iint_{R_\infty}\sum_{i,\,j=1,\,i\neq j}^{n} \frac{g[z_R + i(J_j/J_i)z_I]\Omega^{ij}(x,y,z_R,z_I)\,dz\wedge d\bar{z}}{z-k}, \tag{14}$$

where $\Omega^{ij}(x,y,z_R,z_I) = T_{ij}(z_R,z_I)w^{ij}(x,y,k_R,k_I)$, T_{ij} is part of the scattering data, and w^{ij} is an nth-order matrix with zeros everywhere except at its ijth entry which equals $\exp[\theta_{ij}(x,y,k)]$, $\theta_{ij}(x,y,k) \equiv i(J_i - J_j)(J_i k_R x + k_I y)/J_i$. The matrices $\hat{S}(x,y)$, $\hat{T}(x,y)$, $S(x,y,k)$ have columns

$$\underline{\hat{S}}_{l_i}{}^i(x,y) \equiv \lim_{k\to k_{l_i}{}^i}[\underline{S}^i(x,y,k) - \underline{\varphi}_{l_i}{}^i/(k - k_{l_i}{}^i)],$$

$$\underline{\hat{T}}_{l_i}{}^i(x,y) \equiv \{[T_{x,y,\Omega}\,\mu(x,y,\cdot)](k_{l_i}{}^i)\}_i,$$

$$\underline{S}^i(x,y,k) \equiv \sum_{j=1}^{n}\sum_{l_j=1}^{\Lambda_j}\underline{\varphi}_{l_j}{}^j\exp[-\hat{\theta}_{ij}(x,y,k_{l_j}{}^j)]/[k - (k_{Rl_j}{}^j + ik_{Il_j}{}^j J_j/J_i)].$$

In the above expressions $\{g\}_i$ denotes the ith column of the matrix g and $\hat{\theta}_{ij} \equiv \theta_{ij} + i\sigma_{ij}$, where the constants σ_{ij} are part of the scattering data.

Equations (13) define the matrix $\mu(x,y,k)$ and the vectors $\{\varphi_{l_i}{}^i(x,y)\}_{l_i=1}{}^{\Lambda i}$, $i=1,\ldots,n$, in terms of $\{k_{l_i}{}^i, \gamma_{l_i}{}^i, (\sigma_{ij})_{l_i}{}^i\}_{l_i=1}{}^{\Lambda i}$, $i=1,\ldots,n$ and $T_{ij}(k_R,k_I)$, $i,j=1,\ldots,n$, $i\neq j$. Given this scattering data the potential $q(x,y)$ is reconstructed from

$$q(x,y)=\hat{J}\left[\frac{1}{2\pi}\iint_{R_\infty}\sum_{i,j=1,i\neq j}^{n}\mu\left(x,y,z_R+i\frac{J_j}{J_i}z_I\right)\Omega^{ij}(x,y,z_R,z_I)dz\wedge d\bar{z}-i\hat{\Phi}(x,y)\right], \tag{15}$$

where the ith column of the matrix $\hat{\Phi}(x,y)$ is given by

$$\sum_{j=1}^{n}\sum_{l_j=1}^{\Lambda n}\Phi_{l_j}{}^j(x,y)\exp[-\hat{\theta}_{ij}(x,y,k_{Rl_j}{}^j)].$$

Let $J=\mathrm{diag}(1,-1)$, $q_{12}=Q$, $q_{21}=-\sigma\bar{Q}$, $\sigma=\pm 1$,

$$\Omega^{ij}(x,y,k_R,k_I,t)=\exp\{[k_R+i(J_j/J_i)k_I]^2tA\}\,\Omega^{ij}(x,y,k_R,k_I,0)\exp(-k^2tA),$$

$$A\equiv\mathrm{diag}[i,-i],\quad \partial k_{l_i}{}^i/\partial t=0,\quad \partial\gamma_{l_i}{}^i/\partial t=-2k_i{}^iA_i,$$

and

$$\partial(\sigma_{ji})_{l_i}{}^i/\partial t=A_j\{2ik_{Rl_i}{}^i[1-(J_i/J_j)]^2-[1-(J_i/J_j)]^2(k_{Il_i}{}^i)^2\}.$$

Then Q satisfies DSII, i.e., the equation obtained from (12) by replacing Q_{xx} with $-Q_{xx}$ and φ_{yy} with $-\varphi_{yy}$. Pure lump solutions correspond to $\Omega^{ij}(x,y,k_R,k_I,0)=0$.

We remark that the "$\bar{\partial}$" approach was first used by Beals and Coifman[7] for IST in 1+1 dimensions. However, in 1+1 as opposed to 2+1 dimensions the Riemann-Hilbert approach is sufficient. Moreover, in 2+1 dimensions the "$\bar{\partial}$" approach requires a nontrivial symmetry relationship between $\partial\mu/\partial\bar{k}$ and μ.

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[1]M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Studies in Applied Mathematics Vol. 4 (Society for Industrial and Applied Mathematics, Philadelphia, 1981).

[2]P. D. Lax, Commun. Pure Appl. Math. 21, 467 (1968).

[3]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967), and Commun. Pure Appl. Math. 27, 97 (1974).

[4]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973), and 31, 125 (1973), and Stud. Appl. Math. 53, 249 (1974).

[5]M. J. Ablowitz and R. Haberman, Phys. Rev. Lett. 35, 1185 (1975).

[6]I. M. Gel'fand and L. A. Dikii, Funct. Anal. Appl. 11, 93 (1977).

[7]R. Beals and R. R. Coifman, "Scattering and Inverse Scattering for First Order Systems" (to be published); R. Beals, "The Inverse Problems for Ordinary Differential Operators on the Line" (to be published); R. Beals and R. R. Coifman, Seminaire Goulaoric-Meyer-Schwartz, Ecole Polytechnique, Palaiseau, 1981–1982, exp. 21 (unpublished).

[8]V. E. Zakharov and S. V. Manakov, Sov. Sci. Rev. Phys. Rev. 1, 133 (1979); S. V. Manakov, Physica (Utrecht) 3D, 420 (1981).

[9]D. J. Kaup, Physica (Utrecht) 1D, 45 (1980); H. Cornille, J. Math. Phys. 20, 1653 (1979); L. P. Niznik, Ukr. Mat. J. 24, 110 (1972).

[10]A. S. Fokas and M. J. Ablowitz, Stud. Appl. Math. 68, 1 (1983).

[11]A. S. Fokas and M. J. Ablowitz, Phys. Lett. 94A, 67 (1983), and Institute for Nonlinear Studies Report No. 22, November 1982 (to be published).

[12]M. J. Ablowitz, D. Bar Yaacov, and A. S. Fokas, Institute for Nonlinear Studies Report No. 21, November 1982 (to be published).

[13]A. S. Fokas, preceding Letter [Phys. Rev. Lett. 51, 3 (1983)].

[14]A. S. Fokas and M. J. Ablowitz, Institute for Nonlinear Studies Report No. 24, January 1983 (to be published).

[15]A. Davey and K. Stewartson, Proc. Roy. Soc. London, Ser. A 338, 101 (1974); D. J. Benney and G. J. Roskes, Stud. Appl. Math. 48, 377 (1969).

[16]L. Hörmander, *Complex Analysis in Several Variables* (Van Nostrand, Princeton, N.J., 1966).