Fractional Quantization of the Hall Effect: A Hierarchy of Incompressible Quantum Fluid States

F. D. M. Haldane
Department of Physics, University of Southern California, Los Angeles, California 90089
(Received 28 June 1983)

With use of spherical geometry, a translationally invariant version of Laughlin's proposed "incompressible quantum fluid" state of the two-dimensional electron gas is formulated, and extended to a hierarchy of continued-fraction Landau-level filling factors ν. Observed anomalies at $\nu=\frac{2}{5}, \frac{2}{7}$ are explained by fluids deriving from a $\nu=\frac{1}{3}$ parent.

PACS numbers: 71.45.Nt, 72.20.Nt, 73.40.Lq

The quantum Hall effect (quantization of the Hall resistance $\rho_{xy}=h/\nu e^2$ at simple rational values of ν at low temperatures, together with a dramatic fall in the sheet resistance ρ_{xx}) observed in GaAs-$\mathrm{Ga}_x\mathrm{Al}_{1-x}\mathrm{As}$ heterostructures[1,2] may be explained (naively) if the ground state of the two-dimensional (2D) electron gas in high perpendicular magnetic fields has *no gapless excitations* (and hence no dissipation at low temperatures) when the Landau-level occupation factor takes one of the quantized values ν. This is trivially the case for free electrons when ν is integer, as seen in the earlier experiments,[1] but the effect (or its precursor anomalies) has recently been observed[2] with *fractional* quantization, to date at $\nu=\frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}, \frac{4}{3}$, and $\frac{5}{3}$, all with *odd* denominators (when $\nu>1$, the electrons are not fully spin-polarized; $\nu=\frac{4}{3}, \frac{5}{3}$ values may be understood as the $\nu=1$ effect for majority spins, plus the $\nu=\frac{1}{3}, \frac{2}{3}$ effects for minority spins). A "Wigner solid" charge-density-wave ground state is expected[3] at low occupations, but such a state has a gapless Goldstone mode because translational and rotational symmetry (described by the Euclidean group) is broken. A state without gapless excitations may instead be characterized[4,5] as an "incompressible quantum fluid," and variational wave functions of Jastrow form that describe such states have recently been proposed by Laughlin[4] at occupations $\nu=1/m$, m an odd integer.

The Laughlin wave functions are not translationally invariant, but describe a circular droplet of fluid, which must be confined in an external potential. Laughlin circumvented this problem by formally relating the properties of the fluid to those of the classical 2D one-component plasma, which has a thermodynamic limit, and calculating plasma properties. In this Letter, I describe a variant of Laughlin's scheme with fully translationally invariant wave functions, and extend it to describe a hierarchy of fluid states with occupation factors given by the continued fractions

$$\cfrac{1}{m+\cfrac{\alpha_1}{p_1+\cfrac{\alpha_2}{\ddots+\cfrac{\alpha_n}{p_n}}}}$$

where $m=1,3,5,\ldots$, $\alpha_i=\pm 1$, and $p_i=2,4,6\ldots$; this number will be denoted by $[m,\alpha_1 p_1,\alpha_2 p_2,\ldots,\alpha_n p_n]$, and is a rational with an *odd denominator.* The fluid state at $\nu=[m,p_1,\ldots,p_n]$ cannot occur unless its "parent" state at $\nu=[m,p_1,\ldots,p_{n-1}]$ also occurs; whether or not a given fluid state occurs will depend on the details of the interactions. The experimentally observed anomalies with $\nu<1$ correspond to $[3,2]$, $[3]$, $[3,-2]$, $[1,2,-2]$, $[1,2]$, and $[1,4]$; they all derive from the $m=1$ and $m=3$ hierarchies.

The technical innovation that I make is to place a 2D electron gas of N particles on a *spherical* surface of radius R, in a radial (monopole) magnetic field $B=\hbar S/eR^2$ (>0) where $2S$, the total magnetic flux through the surface in units of the flux quantum $\Phi_0=h/e$, is integral as required by Dirac's monopole quantization condition.[6] This device allows the construction of homogeneous states with finite N; in the limit R, N, and $S\to\infty$, the Euclidean group of the plane is recovered from the rotation group $O^+(3)$ of the sphere.

Single-particle states.—The single-particle Hamiltonian is

$$H=|\vec{\Lambda}|^2/2M=\tfrac{1}{2}\omega_c|\vec{\Lambda}|^2/\hbar S,$$

where M is the effective mass, and $\omega_c=eB/M$ is the cyclotron frequency. $\vec{\Lambda}=\vec{r}\times[-i\hbar\nabla+e\vec{A}(\vec{r})]$ is the dynamical angular momentum; $\nabla\times\vec{A}=B\hat{\Omega}$, $\hat{\Omega}=\vec{r}/R$. $\vec{\Lambda}$ has no component normal to the surface: $\vec{\Lambda}\cdot\hat{\Omega}=\hat{\Omega}\cdot\vec{\Lambda}=0$; its commutation relations are $[\Lambda^\alpha,\Lambda^\beta]=i\hbar\epsilon^{\alpha\beta\gamma}(\Lambda^\gamma-\hbar S\Omega^\gamma)$. The generator of rotations is instead given by $\vec{L}=\vec{\Lambda}+\hbar S\hat{\Omega}$:

$[L^\alpha, X^\beta] = i\hbar\epsilon^{\alpha\beta\gamma}X^\gamma$, $\vec{X} = \vec{L}, \hat{\Omega}$, or $\vec{\Lambda}$; this *has* a normal component* $\vec{L}\cdot\hat{\Omega} = \hat{\Omega}\cdot\vec{L} = \hbar S$. This algebra implies the spectrum $|\vec{L}|^2 = \hbar^2 l(l+1)$, $l = S + n$, $n = 0, 1, 2, \ldots$, and that $2S$ is integral (the Dirac condition[6]); $|\vec{\Lambda}|^2 = |\vec{L}|^2 - \hbar^2 S^2 = \hbar^2\{n(n+1) + (2n+1)S\}$. $\hat{\Omega}$ can be specified by spinor coordinates $u = \cos(\frac{1}{2}\theta)\exp(\frac{1}{2}i\varphi)$, $v = \sin(\frac{1}{2}\theta)\exp(-\frac{1}{2}i\varphi)$: $\hat{\Omega}(u,v) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. To describe the wave functions, I choose the gauge $\vec{A} = (\hbar S/eR) \times \hat{\varphi}\cot\theta$; the singularities at the two poles (each admitting flux $S\Phi_0$) have no physical consequence. The Hilbert space of the lowest Landau level ($l = S$, with energy $\frac{1}{2}\hbar\omega_c$) is spanned by the coherent states $\psi_{(\alpha,\beta)}{}^{(S)}$ defined by $\{\hat{\Omega}(\alpha,\beta)\cdot\vec{L}\}\psi_{(\alpha,\beta)}{}^{(S)} = \hbar S\psi_{(\alpha,\beta)}{}^{(S)}$; these are polynomials in u and v of total degree $2S$:

$$\psi_{(\alpha,\beta)}{}^{(S)}(u,v) = (\alpha^* u + \beta^* v)^{2S}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Within this subspace, the electron may be represented as a spin S, the orientation of which indicates the point on the sphere about which the state is localized.[7] The operator $\vec{L}$ can be written as $L^+ = \hbar u\,\partial/\partial v$, $L^- = \hbar v\,\partial/\partial u$, $L^z = \frac{1}{2}\hbar(u\,\partial/\partial u - v\,\partial/\partial v)$, and $S = \frac{1}{2}(u\,\partial/\partial u + v\,\partial/\partial v)$; u and v may also be represented as independent boson creation operators, and $\partial/\partial u$ and $\partial/\partial v$ as their conjugate destruction operators.

Two-particle states.—The operator $|\vec{L}_1 + \vec{L}_2|^2$ has eigenvalues $\hbar^2 J_{12}(J_{12}+1)$, $J_{12} = 0, 1, \ldots, 2S$; the coherent states with $J_{12} = J$, $\{\hat{\Omega}(\alpha,\beta)\cdot(\vec{L}_1 + \vec{L}_2)\}\psi_{(\alpha,\beta)}{}^{(S,J)} = \hbar J\psi_{(\alpha,\beta)}{}^{(S,J)}$, have wave functions

$$\psi_{(\alpha,\beta)}{}^{(S,J)} = (u_1 v_2 - u_2 v_1)^{2S-J}\prod_{i=1,2}(\alpha^* u_i + \beta^* v_i)^J.$$

Fermi statistics requires that $2S - J_{12}$ be odd, and Bose statistics, that it be even. Note that the factor $u_1 v_2 - u_2 v_1$ commutes with $\vec{L}_1 + \vec{L}_2$. If Π_S is the projection operator on states of the lowest Landau level, the projection on rotationally invariant operators $V(\hat{\Omega}_1\cdot\hat{\Omega}_2)$ (such as the interparticle interaction) can be expanded as

$$\Pi_S V(\hat{\Omega}_1\cdot\hat{\Omega}_2)\Pi_S = \sum_{J=0}^{2S} V_J{}^{(S)} P_J(\vec{L}_1 + \vec{L}_2),$$

where $P_J(L)$ is the projection operator on states with $|\vec{L}|^2 = \hbar^2 J(J+1)$. In particular, $\Pi_S(\hat{\Omega}_1\cdot\hat{\Omega}_2)\Pi_S = \vec{L}_1\cdot\vec{L}_2/\{\hbar(S+1)\}^2$; the smaller the value of $2S - J_{12}$, the smaller the mean separation between the particles, which are precessing about their common center of mass at $\hat{\Omega}(\alpha,\beta)$.

N-particle states.—In the spirit of Laughlin,[4] I discuss the N-particle wave function,

$$\Psi_N{}^{(m)} = \prod_{i<j}(u_i v_j - u_j v_i)^m, \quad S = \tfrac{1}{2}m(N-1).$$

The case $m = 1$ can be alternatively expressed as the antisymmetric Slater determinant describing complete filling of the lowest Landau level, with $N = 2S + 1$. Because $\vec{L}_{\rm tot} = \sum_i \vec{L}_i$ commutes with $u_i v_j - u_j v_i$, $\Psi_N{}^{(m)}$ is explicitly translationally and rotationally invariant on the surface of the sphere: $\vec{L}_{\rm tot}\Psi_N{}^{(m)} = 0$. It is totally antisymmetric (Fermi statistics) for odd m, and symmetric (Bose statistics) for even m. The Laughlin droplet wave functions,[4] centered at $\hat{\Omega}(\alpha,\beta)$, can be recovered by multiplying $\Psi_N{}^{(m)}$ by a factor $\prod_i(\alpha^* u_i + \beta^* v_i)^n$, and taking the limit $n\to\infty$, $R\to\infty$, $R^2/2n = a_0{}^2$, where $a_0 = (\hbar/eB)^2$ is the Larmor radius of the lowest Landau level.

Remarks.—(1) $\Psi_{n=3}{}^{(m)}$ is an exact eigenstate of *any* pair interaction $\sum_{i<j}\{\Pi_S V(\hat{\Omega}_i\cdot\hat{\Omega}_j)\Pi_S\}$, because $J_{12} = J_{23} = J_{31} = S = m$; in the planar geometry, Laughlin's $N = 3$ droplet states are reportedly not exact: Overlaps with numerically calculated exact eigenstates[4] (e.g., 0.994 68 for the Coulomb interaction, $m = 5$) are close to, but *not* exactly, unity. (2) for $N \geq 4$, $m > 1$, $\Psi_N{}^{(m)}$ is *not* an exact eigenstate of a general interaction potential: This would require that it is an exact eigenstate with $J_{ij} = J$ of the angular momentum of any pair of particles. The spectrum of values of J_{ij} contained in $\Psi_N{}^{(m)}$ is easily determined by writing it as the product of three factors (i) involving coordinates i,j only, (ii) involving coordinates $k \neq i,j$ only, and (iii) the cross term $\prod_k(v_k u_i - u_k v_i)^m(v_k u_j - u_k v_j)^m$ which determines J_{ij}: $J_{ij} \leq m(N-2) = 2S - m$. The special character of the states $\Psi_N{}^{(m)}$ is thus that they have no components with $J_{ij} = 2S - m + 2, 2S - m + 4, \ldots \leq 2S$ that would be present in a more general wave function of the appropriate symmetry: The states of closest approach of the pair of particles are suppressed. In particular, when $S = S(N; m) \equiv \frac{1}{2}m(N-1)$, $\Psi_N{}^{(m)}$ may be characterized as the *exact nondegenerate ground state* of the projection-operator interaction potential

$$\Pi_S H_{m,S}{}^{\rm int}\Pi_S = \sum_{i<j}\Big\{\sum_{J>2S-m} P_J(\vec{L}_i + \vec{L}_j)\Big\}.$$

This is essentially a kind of hard-core interaction; $\Psi_N{}^{(m)}$ will thus be a particularly good variational approximation for the ground state of systems with strong repulsion at close separations.

Excited states.—In this geometry, the natural excitation operators, analogous to those suggested by Laughlin,[4] are

$$A_N{}^{\dagger}(\alpha,\beta)=\prod_{i=1}^{N}(\beta u_i-\alpha v_i)\quad(\text{"holes"}),$$

$$A_N(\alpha,\beta)=\prod_{i=1}^{N}\left(\beta^*\frac{\partial}{\partial u_i}-\alpha^*\frac{\partial}{\partial v_i}\right)\quad(\text{"particles"}),$$

which, respectively, increase or decrease the flux quantum number S by $\frac{1}{2}$, and decrease or increase $\hat{\Omega}(\alpha,\beta)\cdot\vec{L}_{\text{tot}}$ by $\frac{1}{2}N\hbar$. The single-excitation states $A_N{}^{\dagger}(\alpha,\beta)\Psi_N{}^{(m)}$ and $A_N(\alpha,\beta)\Psi_N{}^{(m)}$ have $J_{\text{tot}}=\frac{1}{2}N$, and describe defects in the fluid localized around[7] $\hat{\Omega}(\alpha,\beta)$. Since

$$[A_N{}^{\dagger}(\alpha,\beta),A_N{}^{\dagger}(\alpha',\beta')]=[A_N(\alpha,\beta),A_N(\alpha',\beta')]=0,$$

the two-hole and two-particle states are symmetric in the excitation coordinates, and the excitations thus obey Bose statistics {note, however, that $[A_N(\alpha,\beta),\,A_N{}^{\dagger}(\alpha',\beta')]\neq 0$}. A state with $N_p{}^{\text{ex}}$ particle and $N_h{}^{\text{ex}}$ hole excitations has $S=S(N;m)+\frac{1}{2}(N_h{}^{\text{ex}}-N_p{}^{\text{ex}})$; on the other hand, if the system is excited by addition or removal of an electron at fixed magnetic field, the final state has $S=S(N+1;m)\mp\frac{1}{2}m$. The comparison indicates that the hole excitations carry a fractional charge $e^*=+e/m$, and the particles a fractional charge $-e^*$, as proposed by Laughlin.[4] The degeneracy $N+1$ of the single-excitation states supports the same conclusion: In the thermodynamic limit there is one state for each unit $\Phi_m\equiv m\Phi_0=h/e^*$ of magnetic flux through the surface.

Hierarchy of fluid states.—I will assume, following Laughlin,[4] that for some m, the ground state of the 2D electron gas with $S=S(N;m)$ is well represented by the approximate wave function $\Psi_N{}^{(m)}$, and that there is a gap in the excitation spectrum, the lowest-energy excitations being (bound) particle-hole pairs. Consider now a slightly different field strength so that $S=S(N;m)+\frac{1}{2}N^{\text{ex}}$; the low-energy states at this field strength can be considered as deriving from the fluid state $\Psi_N{}^{(m)}$ with an imbalance of particle and hole excitations, $N^{\text{ex}}=N_h{}^{\text{ex}}-N_p{}^{\text{ex}}$. Since there is, by assumption, a gap for making particle-plus-hole excitations, the lowest-energy states will belong to a manifold of purely hole states ($N^{\text{ex}}>0$) or purely particle states ($N^{\text{ex}}<0$), separated by a gap from higher-energy states. If the interaction energy of the *excitations* is small compared to this energy gap, the problem of constructing the collective ground state of the *excitation* fluid is precisely analogous to the original problem of constructing the ground state of the *electron* fluid, but with S replaced by $\frac{1}{2}N$, N replaced by $|N^{\text{ex}}|$, and Fermi statistics replaced by Bose statistics. A Laughlin fluid state of the excitations[8] can be constructed if

$$\tfrac{1}{2}N=S(|N^{\text{ex}}|;p),$$

where p is now *even* (Bose statistics): $p=2,4,6,\ldots$. This leads to $|N^{\text{ex}}|=(N/p)+1$; this second family of fluid states thus can occur at field strengths $S=S(N;m,\pm p)\equiv\frac{1}{2}m(N-1)\pm\frac{1}{2}[(N/p)+1]$, and requires that N be divisible by p. If this fluid state exists, with a sufficiently strong gap, the argument can be iterated by constructing a type-$[|p_1|,p_2]$ fluid state of the excitations of the primary type-$[m]$ electron fluid, and so on; the hierarchical set of equations is

$$S(N;m,p_1,\ldots,p_n)=S(N;m)+\tfrac{1}{2}|N^{\text{ex}}|\operatorname{sgn}(p_1);$$

$$\tfrac{1}{2}N=S(|N^{\text{ex}}|;\,|p_1|,p_2,\ldots,p_n).$$

The filling factor ν is given by $N/2S$ in the thermodynamic limit; the hierarchical equations become

$$\{\nu(m,p_1,\ldots,p_n)\}^{-1}=m+\operatorname{sgn}(p_1)\,\nu(|p_1|,p_2,\ldots p_n),$$

with the solution

$$\nu(m,p_1,\ldots,p_n)=[m,p_1,\ldots,p_n].$$

The charge of the excitations is easily found by determining how many are produced by adding an electron at fixed magnetic field: The result is that if ν is expressed as the rational P/Q, Q is odd, $e^*=e/Q$, and the Hall resistance can be written $\rho_{xy}=\Phi_0/Pe^*$, consistent with Laughlin's "gauge invariance" argument.[9]

The above analysis indicates how "incompressible fluid" states may derive from parent "incompressible fluid" states at simpler rational filling factors ν; the most stable fluid states will correspond to the simplest rationals with small values of m and p_i, where the fluid densities are highest, and hence short-range repulsion effects strongest. What is so far missing is a calculational scheme for the direct determination of *whether* a given fluid state exists for a given interaction potential, e.g., the Coulomb interaction. The new formalism based on a spherical geometry may simplify this task. From a variational viewpoint,[4] the correlation energy of $\Psi_N{}^{(m)}$ and its excitations $A_N{}^{\dagger}(\alpha,\beta)\Psi_N{}^{(m)}$ must be determined; this reduces to (i) the determination of the expansion coefficients $V_J^{(S)}$ of the interaction potential, and (ii) analysis of the wave functions to determine the relative weights of the components with a given pair angular momentum J_{ij}. Progress may be possible in this formalism. Beyond the variational approach, the problem has been reduced to a generalized version of an infinite-coordination Heisenberg problem involving N spin-S objects, and direct numerical calculation of the low-lying energy levels at an increasing sequence of values of N with $S = S(N;m,p_1,\ldots,p_n)$, coupled with a *finite-size scaling* analysis of how the gap behaves as $N\to\infty$, may prove possible at simple rationals. It may be remarked that, in this geometry, the gapless Wigner lattice would also derive from an isotropic state $L_{\rm tot}=0$ as $N\to\infty$: The sphere cannot be tiled with a triangular lattice without introducing *disclination defects*; these will be mobile, and will restore translational and rotational invariance.

This work was brought to conclusion during a stay at the Laboratoire de Physique des Solides, Université Paris-Sud, Orsay, and the hospitality and support of R. Jullien and Professor J. Friedel is gratefully acknowledged. The assistance of M. Kolb in numerical study of the case $N=4$ was invaluable in the development of the above ideas.

[1]K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980); D. C. Tsui and A. C. Gossard, Appl. Phys. Lett. 37, 550 (1981).

[2]D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982); H. L. Stormer *et al.*, Phys. Rev. Lett. 50, 1953 (1983).

[3]L. Bonsall and A. A. Maradudin, Phys. Rev. B 15, 1959 (1977); D. Yoshioka and H. Fukuyama, J. Phys. Soc. Jpn. 47, 394 (1979); D. Yoshioka and P. A. Lee, Phys. Rev. B 27, 4986 (1983); K. Maki and X. Zotos, to be published.

[4]R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).

[5]D. Yoshioka, B. I. Halperin, and P. A. Lee, Phys. Rev. Lett. 50, 1219 (1983).

[6]P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 133, 60 (1931).

[7]The convention used here is that negative-charge coherent single-particle states (electrons, "particles") transform *contragrediently* under rotations, while positive-charge states ("holes") transform *cogrediently* ($\vec{L}$ points *away* from their position).

[8]The possibility of a hole fluid is raised, along with other suggestions such as a fluid of electron pairs, in a discussion of how Laughlin's wave functions might be generalized to other rational values of ν by B. I. Halperin, to be published.

[9]R. B. Laughlin, Phys. Rev. B 23, 5632 (1981).