Absolute Parametric Instabilities in Inhomogeneous Plasmas

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For one-dimensional three-wave parametric instabilities with oppositely directed group velocities for the decay waves in unbounded plasmas, it is known that for uniform coupling amplitude and constant wave-vector mismatch ($d\Sigma k_i/dx$=const) there cannot be an absolute instability. It is shown that this is an exceptional case, and that in general an absolute instability can be obtained in an inhomogeneous plasma with a sufficiently strong pump. A simple criterion is given for the threshold of the absolute instability.

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Parametric decay instabilities have been of considerable interest for thermonuclear fusion applications for a long time[1-9] and the possibility of absolute instability with opposed group velocities for the decay products has been analyzed often. The appropriate coupled-mode equations can be written in unnormalized form as follows:

$$\nu_s a_s + \partial a_s/\partial t + s v_s\, \partial a_s/\partial x = \gamma e^{isF} a_{-s}. \qquad (1)$$

Here s is ± 1, and is written as + or − when used as a subscript to denote the right (+) or left (−) traveling decay wave, with group velocity v_s, attenuation ν_s, and complex action amplitude a_s, such that the action density is $a_s a_s^*$. The complex coupling coefficient is written explicitly in terms of its real amplitude γ and phase sF. The value of F is the integral of the coupling wave-vector sum: $F = \int^x dx' [\boldsymbol{k}_{\text{pump}}(x') + k_+(x') + k_-(x')]$. The second derivative of F is of interest; it is the mismatch wave-vector gradient, which we simply call "mismatch" for brevity.

Coupling-mode analysis[4] shows that if the mismatch is zero somewhere (a rather special case) then absolute instability can obtain, and the waves will grow without limit (until nonlinear effects become important). The simplest case for nonzero mismatch is that of a constant mismatch (k', such that F is $k'x^2/2$), with constant coupling amplitude and no losses. This was analyzed by Rosenbluth,[1] and by others.[2-4] In that case it was shown that the exponential temporal amplitude growth was limited to the Rosenbluth factor $\exp R$, where R is given by $\pi/K' = \pi\gamma^2/k'v_+v_-$ and is the amplification factor for a steady source. [The addition of any loss means that the waves eventually decay as $\exp(-\nu t)$, where ν is given below in Eq. (6).] On the other hand, slablike,[5] Gaussian,[4,6] and Lorentzian[7] coupling-*amplitude* spatial profiles were shown to allow unrestricted growth. The addition of sufficient random[8] or sinusoidal[9] coupling-*phase* components to the Rosenbluth coupling-phase model also caused unlimited growth, even though the mismatch was nowhere zero. The necessary conditions for this growth were unknown. In spite of these odd counterexamples, there has been a tendency, derived from the Rosenbluth model, to discount[4] the possibility of linearly unlimited growth in decay processes (such as convective Raman instability) when the inhomogeneity is such that the mismatch is nowhere zero. In fact, in spite of the Rosenbluth result for constant mismatch, for more general conditions unlimited growth is always possible for a sufficiently strong pump. The constant-mismatch model proves to be a very special one, from which one cannot draw general conclusions.

Typical normalized growth rates from direct integration are shown in Fig. 1, against a normalized coupling-amplitude curvature length, of which there is more discussion later. These results encouraged us to believe that characteristic lengths could be important as curvatures (and not just as half-widths) and that for the same curvature many models would behave in much the same way, at least as far as thresholds were concerned.

Analytic coupling models were investigated with use of the interactive program developed by White[10] for the WKB analysis of the Schrödinger equation with a general analytic potential. If we use the standard normalization [time to a reference γ_0^{-1}, velocity to a reference geometric mean velocity $(v_{+0}v_{-0})^{1/2}$, and hence distance to a reference coupling length $l_c = (v_{+0}v_{-0})^{1/2}/\gamma_0$], take Laplace transforms (it is convenient to define $p_s = p + \nu_s$), drop the initial-value terms because we seek eigenmodes, and use capital letters to denote normalized variables, Eq. (1) becomes

$$P_s a_s + sV_s a_s' = a_{-s} \exp G_s. \qquad (2)$$

We have also defined

$$G_s = \ln(\gamma/\gamma_0) + isF,$$

while the normalized spatial derivative d/dX is denoted by a prime. By direct elimination of a_{-s}, or (more easily) by first writing a_s as $h_s \exp(-s\int dX\,P_s/V_s)$ (to absorb the P_s term) and then eliminating the first-order derivative by changing the dependent variable, we obtain the Schrödinger form for either decay component, with the potential Q_s and new variable A_s:

$$A_s'' + Q_s A_s = 0;\quad A_s = a_s V_s^{1/2}\exp\{-\tfrac{1}{2}[G_s - s\int dX(P_s/V_s - P_{-s}/V_{-s})]\},$$
$$Q_s = \frac{\gamma^2}{\gamma_0^2 V_s V_{-s}} + \frac{1}{2}\left[G_s' - \frac{V_s'}{V_s} + s\left(\frac{P_s}{V_s} + \frac{P_{-s}}{V_{-s}}\right)\right]' - \frac{1}{4}\left[G_s' - \frac{V_s'}{V_s} + s\left(\frac{P_s}{V_s} + \frac{P_{-s}}{V_{-s}}\right)\right]^2. \tag{3}$$

Here we discuss only cases where ν_s and v_s are uniform and hence the product $V_s V_{-s}$ is 1 everywhere, resulting in simpler forms for Q_s and A_s:

$$A_s = a_s V_s^{1/2}\exp\{-\tfrac{1}{2}[G_s - sX(P_s/V_s - P_{-s}/V_{-s})]\}, \tag{4}$$

$$Q_s = \gamma^2/\gamma_0^2 + \tfrac{1}{2}G_s'' - \tfrac{1}{4}[G_s' + s(P_s/V_s + P_{-s}/V_{-s})]^2. \tag{5}$$

One can convert any normalized result, say P_0, for a given coupling with equal velocities and no losses (the simplest case) to the equivalent result for the same coupling but with different velocities and arbitrary damping. The unnormalized equivalent p (whose real part is the growth rate) is given by

$$p = \frac{2(v_+ v_-)^{1/2}}{v_+ + v_-}\gamma_0 P_0 - \left(\frac{\nu_+}{v_+} + \frac{\nu_-}{v_-}\right)\frac{v_+ v_-}{v_+ + v_-} = D\gamma_0 P_0 - \nu. \tag{6}$$

As is well known, when the WKB approximation works, the most unstable mode requires the phase integral between appropriate turning points (zeros of Q_s) to be $\pi/2$, with the relevant anti-Stokes line configuration as sketched in Fig. 2(c). Usually when the eigenfrequency is stable the configuration is as sketched in Fig. 2(a), but somewhat before the threshold is reached, the turning points coalesce as shown in Fig. 2(b). This coalescence of turning points in complex configuration space,[7] which can be considered as the onset of the formation of a potential well deep enough to contain the first unstable mode, is a useful concept closely related to the pole coalescence which has been used before in wave-vector space to obtain instability criteria for other problems.[4,11] In several cases (see Table I) we have been able to obtain simple instability criteria, which proved to be of the form $k'll_c < C$, where l is a coupling *curvature* length. What is particularly gratifying is that a combined amplitude and phase model can also be put into this form, providing that one defines the general curvature l^{-2} from the combined curvatures for γ and F:

$$l^{-2} \equiv l_\gamma^{-2} + l_F^{-2}. \tag{7}$$

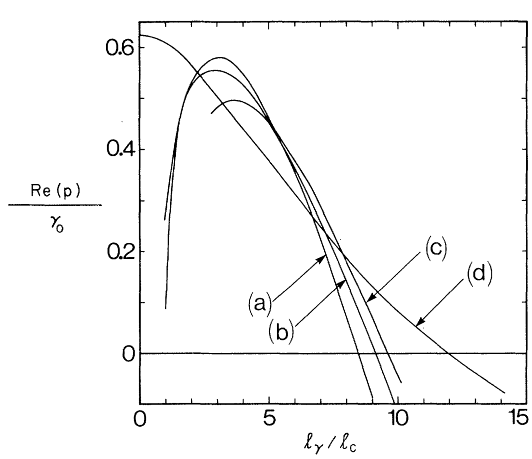

FIG. 1. Normalized growth rates as a function of normalized curvature length for various coupling-amplitude models: curve a, Gaussian; curve b, Lorentzian; curve c, cosine modulation $[0.9\gamma_0 + 0.1\gamma_0\cos(20^{1/2} \times x/l_\gamma)]$; and curve d, half Gaussian, half uniform; all for $k'v_+v_-/\gamma_0^2 = K' = 0.5$.

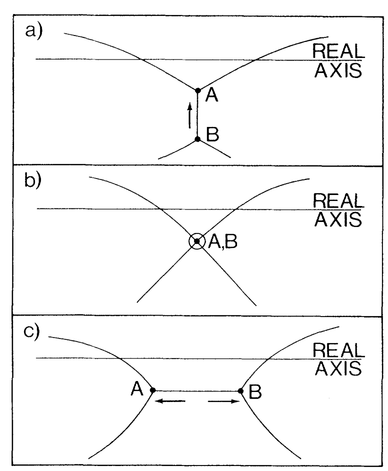

FIG. 2. Turning points in complex X space (a) well below instability, (b) slightly below instability (at coalescence), and (c) at or above threshold.

TABLE I. Instability criteria for several modes of the coupling amplitude and mismatch.

Coupling-amplitude model	Mistmatch model	Instability criterion
Gaussian, $\gamma=\gamma_0 \exp(-x^2/l_\gamma^2)$	Constant, $dF/dx=k'x$	$k'l_\gamma l_c<(8e)^{1/2}$
Lorentzian, $\gamma=\gamma_0/(1+x^2/l_\gamma^2)$	Constant, $dF/dx=k'x$	$k'l_\gamma l_c<3\sqrt{3}$
Constant, $\gamma=\gamma_0$	Quadratic, $dF/dx=k'x(1+x^2/l_F^2)$	$k'l_F l_c<3\sqrt{3}$
Lorentzian, $\gamma=\gamma_0/(1+x^2/l_\gamma^2)$	Quadratic, $dF/dx=k'x(1+x^2/l_F^2)$	$k'll_c<3\sqrt{3}$, $l^{-2}\equiv l_\gamma^{-2}+l_F^{-2}$

This simple formula works well enough, even if the phase-mismatch gradient is not very small, as shown in Fig. 3, which includes $k'v_+v_-/\gamma_0^2=K'$ values as large as 0.5. The criterion has been applied to more intractable analytic models where a simple analytic approximation of the turning-point-coalescence condition is not evident, and still works about as well.

Our combined criterion can be put in a form used earlier for constant mismatch and spatial Gaussian coupling amplitude[4,6] but now using the general length defined above (and correcting a numerical error in an earlier[4] result). Absolute instability requires both

$$k'<l_c^{-2} \text{ and } k'l<3\sqrt{3}\,l_c^{-1}. \tag{8}$$

For practical application this can be written in unnormalized form as a coupling or pump criterion:

$$\gamma_0^2>v_+v_-\max[k',(k'l)^2/27]. \tag{9}$$

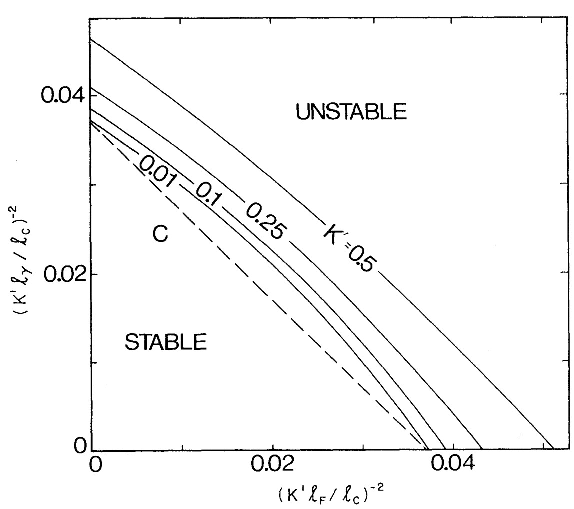

FIG. 3. Instability boundaries for various values of $K'=k'v_+v_-/\gamma_0^2$ with various curvature lengths for γ and F' with a hybrid Lorentzian γ and quadratic mismatch F' as given in Table I. The line C is the locus for the second criterion of Eqs. (8) and (9).

The original Rosenbluth case corresponds to infinite l, and hence to a formally infinite threshold, but for any deviation from that ideal situation, as indicated by a nonzero l^{-1}, there exists a finite threshold for absolute instability.

Note that the ratio of the lossless absolute γ^2 (~ pump intensity) threshold to that required for a particular Rosenbluth convective power amplification $\exp(2R)$ (for which $\gamma^2=v_+v_-k'R/\pi$) is independent of pump strength and is given by $\pi k'l^2/27R$. For typical laser-plasma ultraviolet stimulated Raman-scatter conditions ($n/n_c=0.15$, $\lambda=0.351$ μm, density scale length of 70 μm, $T_e=1$ keV) and with $R=\pi$ (~ a power amplification of 535), this gives a ratio of 60 and hence intensities of order 10^{18} W/cm^{-2}. Unless intensities required for laser fusion increase dramatically this absolute instability should not hamper laser fusion. Magnetic-fusion applications require consideration of particular cases.

This work began with a considerable amount of direct numerical integration of Eq. (1) (whose details will be reported elsewhere). We thank our collaborators in that effort, V. Fuchs, M. Shoucri, and J. P. Matte. The authors also thank in particular R. B. White for furnishing us with a copy of his interactive WKB Schrödinger code. One of us (T.W.J.) acknowledges the hospitality of the University of Rochester Laboratory for Laser Energetics (where part of this work was done), the support of R. L. McCrory, and illuminating discussions with E. A. Williams. Financial support is acknowledged from the Natural Sciences and Engineering Research Council, Canada Team Grant No. A5652.

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