Faddeev-Popov Zeros and Confinement of Color in a Hyperspherical Gauge Model

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A new rotationally invariant Hamiltonian method, formulated on a four-dimensional hyperspherical surface, is proposed for the numerical study of quantum gauge field models. It is shown that in the Coulomb gauge it is sufficient, as well as necessary, to restrict transverse potentials to the zero-free domain of the Faddeev-Popov determinant. Numerical studies of an SU(2) model support Gribov's suggestion that the zeros provide a natural way to understand the origin of confinement.

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The numerical study of lattice gauge models[1,2] has provided information about the origin of color confinement[3] and even about the masses of physical particles.[4,5] However, introduction of fermions is not straightforward, and it is uncertain how big the lattice should be in order to give reliable results.[6] It is therefore useful to consider alternative approaches, which may give complementary information. Direct numerical solution of the Schrödinger equation is feasible if the number of degrees of freedom is limited through use of a limited spatial domain and a short-wavelength cutoff. Here, the spatial domain is taken to be the surface of a four-dimensional hypersphere. Adler's Lagrangian formulation of massless electrodynamics,[7] in which space-time is represented by the surface of a five-sphere, has many useful similarities. The hyperspherical domain has several advantages: The calculations have manifest rotational invariance in all levels of approximation, O(4) group theory can be used to evaluate matrix elements, and the distinction between longitudinal and transverse fields has a simple group-theoretical interpretation.

The existence and importance of zeros of the Faddeev-Popov determinant were pointed out by Gribov,[8] who emphasized that the zeros give a constraint on field variables which provides a confinement mechanism. In the Hamiltonian approach, Gribov's constraint leads to a picture of confinement which is complementary to the Wilson-loop[1] scenario of the thermal-lattice method.[3] A numerical study of the zeros is carried out in this work. This study indicates that the longitudinal and transverse propagators do have forms consistent with confinement. With use of a simple "model determinant" and other approximations similar to those of Gribov,[8] some low-lying states are then examined; the lowest excitation has a finite mass and a finite size.

In the Coulomb gauge, the Hamiltonian is[8,9]

$$H=\tfrac{1}{2}\int d\Omega\{F^{-1}\vec{\mathrm{E}}^{\alpha}\cdot F\vec{\mathrm{E}}^{\alpha}+\vec{\mathrm{B}}^{\alpha}\cdot\vec{\mathrm{B}}^{\alpha}\}$$
$$+\tfrac{1}{2}\int d\Omega\int d\Omega' F^{-1}\zeta^{\beta}(\vec{\mathrm{r}}')FC^{\beta\alpha}(\vec{\mathrm{r}}',\vec{\mathrm{r}})\zeta^{\alpha}(\vec{\mathrm{r}}),$$

where $\zeta^{\alpha}=-f^{\alpha\beta\gamma}\vec{A}^{\beta}\cdot\vec{E}^{\gamma}$, and $\vec{A}$, $\vec{E}$, and $\vec{B}$ are the usual transverse potentials and fields. The longitudinal propagator, which gives the Coulomb potential, is $C=k^{-1}(1-D)^{-2}k^{-1}$, where $k=(-\nabla^2)^{1/2}$, $D^{\alpha\beta}=-k^{-1}gf^{\alpha\beta\gamma}\vec{A}^{\gamma}\cdot\vec{\nabla}k^{-1}$, and the Faddeev-Popov determinant is $F=\det(1-D)$. Hereafter, color indices will be omitted.

To transcribe the Hamiltonian onto a hypersphere, it is necessary to consider scalar and vector functions of the unit vector $\hat{r}$, and also properties of derivative operators on the hypersphere. A unit radius is used, because there is no other scale in this model. It is convenient to use a four-dimensional notation, with the constraint that all vectors should be tangent to the hypersphere. Although the general properties of O(4) are widely known, and many detailed formulas are also available,[10] most of the specific results needed here have not been discussed explicitly. It is useful to employ tensor methods, and also, especially for evaluating matrix elements, the SU(2)⊗SU(2) decomposition. Scalar harmonics $S^{a_1\cdots a_s}$ correspond to the representation $(s/2, s/2)$, while vector harmonics $\vec{V}_b{}^{a_1\cdots a_v}$ correspond to the pair $((v+1)/2,(v-1)/2)$ and $((v-1)/2,(v+1)/2)$, which have opposite handedness. Since both $\hat{r}\cdot\vec{\nabla}$ and $\vec{\nabla}\cdot\vec{V}$ are functions only of $\hat{r}$, and must be antisymmetric in b and any of the a's, they both vanish. Thus $\vec{V}$ is tangent to the hypersphere and satisfies the transversality condition; the set of orthonormal vector functions $\vec{V}$ provides the correct basis for expansion of the transverse fields: $\vec{A}=\sum_{vn}q_{vn}\vec{V}_{vn}$. The sum over v is cut off at a value Λ. Likewise, the S's provide the basis for the operators C and D,

which is also cut off at $s=\Lambda$. With inclusion of the SU(2) color degeneracy factor, the dimensionalities of the scalar and vector representations are respectively $M_s=3(s+1)^2$ and $N_v=6v(v+2)$. On the hypersphere, the ∇^2 operator becomes simply the angular part of the ordinary four-dimensional Laplacian, leading to $k(s)^2=s(s+2)$. Similarly, it can be shown that $\mathrm{curl}\vec{V}_v=\pm(v+1)\vec{V}_v$, where the sign defines the handedness.

Note that $k(s)$ vanishes for $s=0$. To avoid having an infinite energy, the total color charge ζ of physical states must vanish. When this constraint is satisfied, the $s=0$ mode can be dropped, because the gradient operator which appears in D insures that there is no coupling between longitudinal modes with $s>0$ and the mode $s=0$.

Following Gribov's work, there have been several investigations of the Faddeev-Popov zeros.[11,12] It is understood that the transverse potentials must be limited to a connected domain G whose boundary is the first zero encountered as the potentials are increased. Zwanziger[12] has shown that this domain G is convex and bounded. The zeros of F correspond to eigenvectors of D which have a unit eigenvalue. Since D is a linear function of the q's, the eigenvalues d of D for some fixed q_{vn} are inversely proportional to the relative distances from the origin to all the various zeros, for q_{vn} which have the same ratios.

In general gauges, it is not known whether restriction of the field variables q to the domain G is sufficient as well as necessary.[8,12] That is, it has not been proved that each distinct point of G corresponds to a distinct physical situation. In the Coulomb gauge, however, it suffices to show that each distinct point of G leads to a distinct longitudinal propagator C, because then some distributions of external color charges will give different energies. The matrix C is determined by its eigenvectors, which are the same as those of D except for the simple weighting factors $k(s)$, and by the eigenvalues $c=(1-d)^{-2}$. Within G, all eigenvalues satisfy $d<1$, and therefore are uniquely related to the eigenvalues c. The q's are linearly independent, and determine D uniquely. Within G, therefore, C is uniquely related to the q's. This argument fails if one goes outside of G, because then there will be some $d>1$.

A quantitative study of the Faddeev-Popov determinant F, by itself, can give useful information about the structure of the vacuum. Gribov pointed out that the amplitudes of long-wavelength modes (small v) should be constrained to very small values by the zeros of F, while the short-wavelength modes should be unaffected. According to asymptotic freedom ideas, the free-field behavior $\langle q^2\rangle=1/2(v+1)$ should be found for large v. To check these points, the first step is to see how the distance to the nearest zero depends on the ratios of the q's.

Let $Q_v^2=\sum_n q_{vn}^2$, where here and later the index n includes the color variable. For exploration of the properties of F, randomly oriented N_v-component vectors q_{vn} have been chosen, with fixed lengths: $Q_v^2=\langle Q_v^2\rangle$, where $\langle Q_v^2\rangle$ is an iteratively determined self-consistent mean value. A narrow-band approximation (NBA) is used, in which $F\Psi^2$ is assumed to be concentrated near $\langle Q_v^2\rangle$. The largest eigenvalue d determines the first zero, and a first-order perturbation calculation determines how the zero shifts if the q_{vn} are changed slightly. The results of these calculations can be summarized by a simple approximate formula for the locus of the zero: $Z(q)=1-\alpha\sum_v U_v Q_v^2/N_v=0$, where $\alpha=g^2/4\pi$, and where the U_v depend weakly on the q_{vn}, and have the approximate value $\frac{1}{2}$. More precisely, the U_v differ by about 10%–25%, being smallest for $v=\Lambda$. The constraint this gives on the q_{vn} for small v is consistent with Gribov's estimates and remarks, and also shows directly that the transverse propagator is damped for large spacelike separations, although the precise amount of damping, and the behavior for timelike separations, require finding the eigenstates of H.

The distance to the nearest zero fluctuates around the value given by $Z(q)$ by about 3%, for $\Lambda=4$; these fluctuations decrease as Λ increases. The average separation between the first and second zeros, and between the second and third, is about 6% for $\Lambda=4$, and also decreases with increasing Λ.

The quantity

$$R(s)=k(s)^2\sum_m C_{sm,sm}/M_s\,\mathrm{Tr}(C),$$

averaged over the q's, gives the ratio of the longitudinal propagator to the free-field propagator. Gribov conjectured that fields very near the boundary of G dominate the vacuum state; this suggests the approximation $R_0(s)=\sum_m(\varphi_{sm})^2/M_s$, where φ is the eigenvector for the first zero. Values of $R_0(s)$ are plotted versus $k(s)^2$ in Fig. 1. These values have a sampling error of a few percent. The α dependence is a secondary effect, and in comparison is not significant. A linear potential would correspond to a unit slope on this graph; the slope for R_0 is actually some-

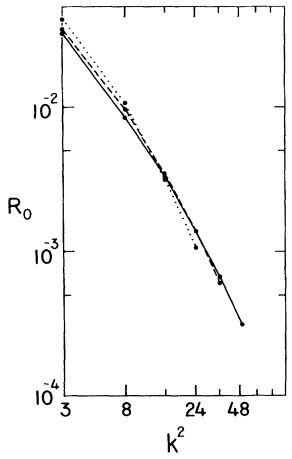

FIG. 1. $R_0(s)$ vs $k^2(s)$ for $\alpha = 5$. The points are connected by a solid line for $\Lambda = 6$, by a dashed line for $\Lambda = 5$, and by a dotted line for $\Lambda = 4$.

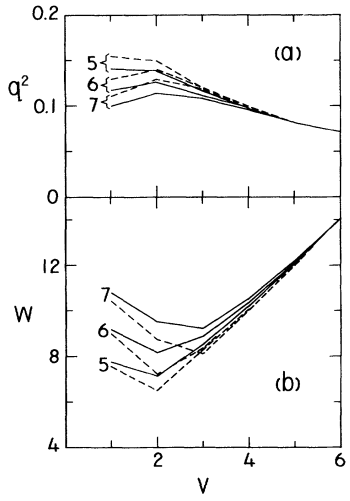

FIG. 2. Results of calculations for $\Lambda = 5$ (dashed lines) and $\Lambda = 6$ (solid lines) for $\alpha = 5$, 6, and 7 as indicated. (a) Mean value $\langle q_v{}^2 \rangle$; (b) excitation energy W_v for $\lambda = 0$.

what greater. The second and third eigenvectors give very similar values for $R_0(s)$, and so this behavior should be obtained over a sizable region near the boundary.

The next step is to calculate F in the self-consistent band $Q_v{}^2 = \langle Q_v{}^2 \rangle$. At the same time, the inverse of $1 - D$ is calculated, giving values for the derivatives of F. A reasonably good approximation to F in the NBA is given by the following simple formula:

$$F_{\text{approx}} = f_0 Z(q) \exp(-\textstyle\sum_v \beta_v Q_v{}^2),$$

where f_0 is a constant and the β_v are chosen to reproduce the correct derivatives. The exponential factor represents the effect of more distant zeros in a form which is tractable for subsequent numerical calculations. The coefficients β_v drop off with increasing v, but not as rapidly as the factors U_v/N_v in $Z(q)$.

Following Gribov,[8] for an initial examination of the properties of low-lying states the charge fluctuations are neglected, along with the self-coupling terms in B. Hartree-like wave functions of the form $\Psi = \prod_v \psi_v(q_{vn})$ are used. The equations for ψ_v are decoupled and involve an averaged determinant for the vth sector:

$$F_v(Q_v{}^2) = f_{0v} z_v(Q_v{}^2) \exp(-\beta_v Q_v{}^2).$$

In the NBA, the effective zero factor takes the form $z_v(Q_v{}^2) = 1 - Q_v{}^2/Q_{0v}{}^2$. The Schrödinger equation for ψ_v has $O(N_v)$ symmetry, and the charge operator in the vth sector is an $O(N_v)$ generator. Along with the ground state and the radial excitations ($\lambda = 0$), only certain excitations with $\lambda \geq 2$ avoid having a net color charge, where λ is the $O(N_v)$ orbital quantum number.

An adequate estimate for the energy of orbital excitation is $W_v = \lambda(\lambda + N_v - 2)/2\langle Q_v{}^2 \rangle$. For $\lambda = 2$, this becomes $W_v = 1/\langle q_{vn}{}^2 \rangle$. The energies of radial excitations, which correspond to helicity-0 particles at rest, are harder to calculate. If the zero-factor z_v is dropped, the equation for ψ_v has a harmonic-oscillator form, giving the energies $E_{v,n} = n\omega_v + N_v(\omega_v - \beta_v)/2$, or $W_v = 2\omega_v$, where $\omega_v{}^2 = (v+1)^2 + \beta_v{}^2$. At $Q_v = Q_{0v}$, the free boundary condition $F_v \psi_v{}' = 0$ is used (this differs from Gribov's treatment). If Q_{0v} is small, ψ_v will be nearly constant in the ground state, but must vary rapidly if there are nodes. Thus, both factors in F act to decrease the electric and magnetic energies in the vacuum state, and to increase the electric energy in excited states. This is consistent with the idea that asymptotic freedom and confinement are associated with diaelectric and paramagnetic[13] properties of the vacuum. For numerical calculations, it is convenient to use trial wave functions of the harmonic-oscillator type. This gives integrals which can be expressed in terms of confluent hypergeometric functions. Some results are displayed in Fig. 2.

In Fig. 2(a) the mean value $\langle q_{vn}{}^2 \rangle$ is plotted for several values of α and $\Lambda = 5, 6$. This shows the reduction of $\langle q_{vn}{}^2 \rangle$ for small v, and the approach to the free-field behavior for large v. The energy needed to excite the vth $\lambda = 0$ mode, W_v, is plotted in Fig. 2(b) for the same parameter values. The excitation energies for $\lambda = 2$, as esti-

mated by $1/\langle q_{vn}{}^2\rangle$, are similar. For large v, the free-field energy $W_v = 2(v+1)$ is obtained, but for these values of α, the minimum occurs at an intermediate value of v and is greater than the free-field minimum energy $W_1 = 4$. As expected, the minimum is raised and shifted to a larger v when α is increased, indicating that the confining potential has been stiffened. Both parts of Fig. 2 show a sizable difference between $\Lambda = 5$ and $\Lambda = 6$. This is an illustration of the antiscreening effect; the effective coupling constant is substantially larger when Λ is increased.

All features of these calculations support the principle of color confinement. Although some of Gribov's approximations and assumptions have been dropped or modified, his main conclusions have been confirmed. For more quantitative results, in particular to estimate the string tension and the glueball spectrum, it will be necessary to examine carefully the remaining approximations.

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