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# Absence of Long-Range Order above Two Dimensions

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It is shown that a $d$-dimensional statistical system of a single U(1) variable, $\exp(i\varphi)$, whose Hamiltonian is invariant under the transformation $\varphi(x_1,\ldots,x_d)\to\varphi(x_1,\ldots,x_d)+\Lambda(x_3,\ldots,x_d)$, with $\Lambda$ an arbitrary function, has no long-range order, so that $\langle\exp(i\varphi)\rangle=0$ for all nonzero temperatures. Moreover, the full planar symmetry reflected in the above transformation law is also unbroken for all $T>0$. When $d=2$ the usual Mermin-Wagner result is recovered. Various extensions and physical implications of this theorem are briefly discussed.

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The most usual types of statistical theories are either those with a simple global symmetry, such as the Ising or O($N$) Heisenberg models, or those with local gauge symmetries. Between these two extremes, however, are theories with Hamiltonians which are invariant under a symmetry transformation expressed by a gauge function which is an arbitrary function of only a subset of the spatial coordinates of the system. If, for a $d$-dimensional theory the gauge function is an arbitrary function of only $d-n$ coordinates, we will say that the theory has an $n$-dimensional symmetry.

Of particular interest is the case $n=2$. For $d=2$ this case corresponds to the usual class of globally symmetric two-dimensional spin systems. For $d=3$, several statistical models with $n=2$ have been studied in the literature.[1,2] When these models are endowed with a continuous symmetry, they show an absence of long-range order, as well as a number of other intriguing properties. Furthermore, such three-dimensional models may well correspond to certain helical magnetic or liquid-crystal systems, in which there is an absence of long-range order for all $T>0$ for certain regions of the phase diagram.

Unlike the two-dimensional case, where the global symmetry expresses the full content of the $n=2$ symmetry, the global symmetry of a theory with $n=2$ and $d\geqslant 3$ is just a subset of the full $n=2$ symmetry corresponding to $n'(\text{global})=d$. In view of the central role of global symmetry breaking in statistical physics, it is important to address the possibility that the full $n=2$ symmetry may be broken without breaking the global symmetry.[3] In this Letter we will show that for $d\geqslant 2$ all theories which are theories of a single U(1) spin, $\exp(i\varphi)$, which have $n=2$ have no long-range order (i.e., no spontaneously broken global symmetry), for any nonzero temperature. Moreover, we will show that for $d\geqslant 3$ the full $n=2$ symmetry is also not broken for any $T>0$. Of course, just as in the case of the two-dimensional $x$-$y$ model, these theories may have phase transitions despite the absence of symmetry breaking. Indeed, some three-dimensional models with $n=2$ have been analyzed and found to undergo a phase transition into a low-temperature phase with no long-range order.[4]

The proof involves three steps. First we will generalize the framework of the usual proof of the Mermin-Wagner theorem[5] to accommodate the larger class of theories in which we are interested. Next we will argue that the existence of the $n=2$ symmetry implies the existence of a $(d-2)$-dimensional surface of singularities in the propagator for the spin waves of the theory. Finally, we will show that, as a result of these singularities, a certain integral diverges with the size of the system, and, as in the proof of the Mermin-Wagner theorem for two-dimensional theories, this divergence implies that the symmetries (global and full $n=2$) are not spontaneously broken for $T>0$. The Letter will conclude with a few ancillary comments.

Consider a theory with a Hamiltonian, $H(\{\varphi\})$, where $\varphi(\vec{x})$ is an angle-valued variable associated with a lattice site with coordinate $\vec{x}$. For simplicity we will take our theory to be defined on a $d$-dimensional hypercubic lattice, but this restriction is not essential for the proof. Following the authors of Ref. 5 we will use the Bogoliubov inequality[6]

$$\tfrac{1}{2}\langle\{A,A^\dagger\}\rangle\langle[[C,H],C^\dagger]\rangle \geq k_B T|\langle[C,A]\rangle|^2. \quad (1)$$

$A^\dagger$ ($C^\dagger$) is the Hermitian conjugate of $A$ ($C$). $C$ is defined by

$$C_k|\varphi(\vec{x})\rangle = |\varphi(\vec{x}) - \delta\varphi\cos\vec{k}\cdot\vec{x}\rangle + |\varphi(\vec{x}) - \delta\varphi\sin\vec{k}\cdot\vec{x}\rangle. \quad (2)$$

The state $|\varphi(\vec{x})\rangle$ is the state defined by the set $\{\varphi(\vec{x})\}$ for all points, $\vec{x}$, on the lattice. A commutator is denoted by $[\,,\,]$, $\{\,,\,\}$ is an anticommutator, $\delta\varphi$ is a small constant field, and

$$\langle O\rangle = \mathrm{Tr}\,O e^{-\beta H}/\mathrm{Tr}\,e^{-\beta H},$$

with

$$\beta = (k_B T)^{-1}.$$

To study the two cases of global and $n=2$ symmetries we need two different sets of $A$ operators:

*Global symmetry.*—

$$A_k|\varphi(\vec{x})\rangle = \textstyle\sum_{\vec{y}} \cos\vec{k}\cdot\vec{y}\,\sin\varphi(\vec{y})|\Phi_1(\vec{x})\rangle + \sum_{\vec{y}} \sin\vec{k}\cdot\vec{y}\,\sin\varphi(\vec{y})|\Phi_2(\vec{x})\rangle. \quad (3)$$

*$n=2$ symmetry.*—

$$A_k|\varphi(\vec{x})\rangle = \textstyle\sum_{\vec{y}}[\cos\vec{k}\cdot\vec{y} - \cos\vec{k}\cdot(\vec{y}-\vec{M})]\Delta_M(\varphi(\vec{y}))|\Phi_1(\vec{x})\rangle + \sum_{\vec{y}}[\sin\vec{k}\cdot\vec{y} - \sin\vec{k}\cdot(\vec{y}-\vec{M})]\Delta_M(\varphi(\vec{y}))|\Phi_2(\vec{x})\rangle, \quad (4)$$

where

$$|\Phi_1(\vec{x})\rangle \equiv |\varphi(\vec{x}) + \delta\varphi\cos\vec{k}\cdot\vec{x}\rangle, \quad |\Phi_2(\vec{x})\rangle \equiv |\varphi(\vec{x}) + \delta\varphi\sin\vec{k}\cdot\vec{x}\rangle, \quad \Delta_M(\varphi(\vec{y})) \equiv \sin[\varphi(\vec{y}) - \varphi(\vec{y}+\vec{M})].$$

In Eq. (4), $\vec{M}$ is a fixed vector with a nonzero projection out of the plane of the $n=2$ symmetry; e.g., if the gauge function $\Lambda$ is independent of $x_1$ and $x_2$ and $\vec{L} = (0,0,1,1,\ldots,1)$, then $\vec{L}\cdot\vec{M} \neq 0$.

Let us first derive the condition for the absence of a spontaneous breakdown of global symmetry. To do this we consider the Hamiltonian of the system in an external magnetic field $h$:

$$H = \textstyle\sum_{\vec{x}} H_0(\varphi(\vec{x})) - h\sum_{\vec{x}} \cos\varphi(\vec{x}). \quad (5)$$

We assume that $H_0$ can be written in the form

$$\sum_{\vec{x}} H_0(\varphi(\vec{x})) = \sum_{\vec{x}}\sum_{p=1}^{s} f_p(\Omega_p(\{\varphi(\vec{x})\})), \quad (6)$$

where

$$\Omega_p(\{\varphi(\vec{x})\}) = \sum_{j=1}^{q(p)} c_{pj}\varphi(\vec{x}_j).$$

$H_0$ contains $s$ different kinds of interactions. $\Omega_p(\{\varphi(\vec{x})\})$ is a linear combination of $\varphi$'s on lattice sites in the neighborhood of some point $\vec{x}$; $\vec{x}_j = \vec{x} + \vec{r}_j$. There are no explicit long-range forces, so that $|\vec{r}_j|$ is finite. We assume further that $f_p$ can be expanded in a Taylor series about the zero of its argument[7]:

$$f(z) = f(0) + f'(0)z + \tfrac{1}{2}f''(0)z^2 + \ldots.$$

We will now use (2) and (3) to calculate (1), expanding in powers of $\delta\varphi$. For small $\delta\varphi$ we have

$$\langle [C_k, A_k] \rangle = \delta\varphi \langle \textstyle\sum_{\vec{x}} \cos\varphi(\vec{x}) \rangle = mN\delta\varphi, \tag{7}$$

where $N$ is the number of lattice sites and $m$ is the magnetization. Furthermore,

$$\sum_{\vec{k}} \tfrac{1}{2}\langle \{A_{\vec{k}}, A_{\vec{k}}^\dagger\}\rangle \le \tfrac{1}{2}\sum_{\vec{k}} \sum_{\vec{x},\vec{x}'} \cos\vec{k}\cdot(\vec{x}-\vec{x}') \le N^2. \tag{8}$$

Finally,

$$\langle \varphi(\vec{x}) | [[C_{\vec{k}}, H], C_{\vec{k}}^\dagger] | \varphi(\vec{x})\rangle = (\delta\varphi)^2 \sum_{\vec{x}} \{h\cos\varphi(\vec{x}) + \sum_{p=1}^{s} f_p''(\Omega_p) \sum_{i,j=1}^{q(p)} c_{pi} c_{pj} \cos\vec{k}\cdot(\vec{r}_i - \vec{r}_j)\}, \tag{9}$$

where $\{\vec{r}_j\}$ is the set of $q(p)$ $\vec{r}$'s defined after Eq. (6)

We now assume that the thermal average, $\langle f_p''(\Omega_p(\{\varphi(\vec{x})\}))\rangle \le \gamma$ for each $p$, where $\gamma$ is some positive number. Then, taking the thermal average of (9), we have

$$\langle [[C_k, H], C_k^\dagger]\rangle \le (\delta\varphi)^2 N[hm + \gamma \sum_p \sum_{i,j=1}^{q(p)} c_{pi} c_{pj} \cos\vec{k}\cdot(\vec{r}_i - \vec{r}_j)]. \tag{10}$$

Using (7), (8), and (10) in (1), we finally obtain

$$m^2 \le \left(\frac{k_B T}{N} \sum_{\vec{k}} \frac{1}{\gamma \sum_p \epsilon_p + hm}\right)^{-1}. \tag{11}$$

where the sum over $\vec{k}$ is over the first Brillouin zone, and

$$\epsilon_p = \left(\sum_{j=1}^{q(p)} c_{pj} \cos\vec{k}\cdot\vec{r}_j\right)^2 + \left(\sum_{j=1}^{q(p)} c_{pj} \sin\vec{k}\cdot\vec{r}_j\right)^2. \tag{12}$$

We will now show that the integral (sum over $\vec{k}$) on the right-hand side of (11) diverges as $h \to 0$ in the thermodynamic limit. First we note that if $H_0$ is invariant under

$$\varphi(x_1, \ldots, x_d) \to \varphi(x_1, \ldots, x_d) + \Lambda(x_3, \ldots, x_d), \tag{13}$$

then $H_0$ can be written in the form of Eq. (6) in which each of the $f_p$ is invariant under (13). This means that the $c_{pj}$ in Eq. (6) must satisfy

$$\sum_{j\in\alpha} c_{pj} = 0, \tag{14}$$

where the set $\alpha$ includes all those spin labels that lie in any given $(x_1, x_2)$ plane. (In continuum notation this is equivalent to the statement that $H_0$ is a function of $\varphi$ only through $\partial_1\varphi$ and $\partial_2\varphi$.) With use of (14), it is clear that, for $k_1 = k_2 = 0$, $\epsilon_p = 0$. Moreover, $\epsilon_p$ can obviously be expanded in a power series in $k_1$ and $k_2$, so that, in general, we can write

$$\epsilon_p = k_1^2 u_p(\vec{k}) + k_2^2 v_p(\vec{k}) + k_1 k_2 w_p(\vec{k}), \tag{15}$$

where $u_p$, $v_p$, and $w_p$ are finite functions of $\vec{k}$.

We consider now a large lattice of linear dimension $L$. Inserting (15) into (11), converting the sum over $\vec{k}$ into an integral in the usual way, and defining $u(\vec{k}) = \gamma\sum u_p(\vec{k})$, etc., we have

$$m^2 \le \left(\frac{k_B T}{(2\pi)^d} \int_{1/L} d^d k \frac{1}{k_1^2 u(\vec{k}) + k_2^2 v(\vec{k}) + k_1 k_2 w(\vec{k}) + hm}\right)^{-1}. \tag{16}$$

From (12), it is clear that $\epsilon_p \ge 0$. Therefore, focusing attention on the region of integration where $k_1$ and $k_2$ are close to $1/L$, it is easy to see that, when $h \to 0$, the integral in (16) diverges at least as fast as $\ln L$. Hence, in the thermodynamic limit, $m^2 \to 0$ as $h \to 0$, and the system has no long-range order.

To prove that there is no spontaneous breaking of the full $n = 2$ symmetry, we follow nearly the same path that led to (16) with the following changes: Instead of expression (3) we use expression (4) in the Bogoliubov inequality, (1), and instead of the global-symmetry-breaking term of Eq. (5), we add to $H_0$ a term which respects the global symmetry but breaks the $n = 2$ symmetry. We thus consider the Hamiltonian

$$H = \textstyle\sum_{\vec{x}} H_0(\varphi(x)) - h_M \sum_{\vec{x}} \cos[\varphi(\vec{x}) - \varphi(\vec{x} + \vec{M})]. \tag{17}$$

With these changes it is straightforward to compute the inequality analogous to (16) which will yield a bound on the breaking of the $n=2$ symmetry. The result is

$$m_M^{\,2} \leq \left( \frac{4k_B T}{(2\pi)^d} \int_{1/L} d^d k \, \frac{\xi_M^{\,2}(\vec{k})}{k_1^2 u(\vec{k}) + k_2^2 v(\vec{k}) + k_1 k_2 w(\vec{k}) + \xi_M(\vec{k}) h_M m_M} \right)^{-1}, \quad \xi_M(\vec{k}) = 1 - \cos\vec{k}\cdot\vec{M}, \tag{18}$$

where

$$m_M = \langle \cos[\varphi(\vec{x}) - \varphi(\vec{x}+\vec{M})] \rangle. \tag{19}$$

Since $\vec{M}$ has a component out of the $(x_1, x_2)$ plane, the numerator of the integrand in (18) does not vanish when $k_1 = k_2 = 0$. Because the integrand in (18) is positive, the analysis proceeds as for Eq. (16). The integral diverges at least as fast as $\ln L$ in the limit $h_M \to 0$, because of the contribution near $k_1 = k_2 = 0$. Thus, $m_M^{\,2} = 0$ in the thermodynamic limit and the $n=2$ symmetry is not spontaneously broken.

We conclude with a few remarks. First, our result is really more general than we have stated it. The condition that the otherwise arbitrary function $\Lambda$ be independent of $x_1$ and $x_2$ is only one of a variety of constraints on $\Lambda$ that will allow us to prove an absence of long-range order. A quick review of the argument leading to Eq. (15) indicates that a large class of conditions on $\Lambda$ of the form $O_1\Lambda = O_2\Lambda = 0$, where $O_1$ and $O_2$ are linearly independent differential operators (or their lattice equivalents), will be sufficient to produce a form for $\epsilon_p$ which vanishes fast enough as two of the components of $\vec{k}$ go to zero to ensure that the integral in (16) diverges for $h \to 0$ and $L \to \infty$. This is the type of condition obeyed by the gauge function in the theory discussed in Ref. 1. Second, it is clear from the derivation of expression (16) that a $d$-dimensional U(1)-invariant theory with an $n=1$ symmetry will also lack long-range order. In this case, the corresponding integral on the right-hand side of (16) will diverge at least as fast as $L$. The absence of long-range order in such a system can be thought of as having a somewhat trivial origin in that the model can be decoupled into a stack of $(d-1)$-dimensional mutually noninteracting models. This is quite analogous to what happens in a genuine one-dimensional model with finite-range interactions. Finally, we remark that although we have restricted ourselves to systems with a U(1) symmetry, we expect, on general grounds, that an analogous theorem will hold for a large class of suitably defined $d$-dimensional, continuously symmetric non-Abelian theories with $n=1$ or $n=2$.

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[3]Actually, one could contemplate breaking any of the $n'$ symmetries, where $d \geq n' > n$, which are subsets of the full $n$-dimensional symmetry without breaking the $n''$ symmetries with $n'' > n'$. As we shall see, without explicit symmetry-breaking fields, this possibility does not exist for the case $n = 2$.

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[7]In a continuum theory these conditions on $H_0$ would be replaced by the condition that $H_0$ be expandable in a power series in $\varphi$ and all possible combinations of a finite number of derivatives. That is, terms like $\varphi|\partial\varphi|$ would not be allowed.