Formation of Saturated Solitons in a Nonlinear Dispersive System with Instability and Dissipation

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A simple one-dimensional nonlinear equation including effects of instability, dissipation, and dispersion is examined numerically. It is observed that for the strongly dispersive case the temporal evolution is characterized by formation of a row of solitary pulses of equal equilibrium amplitudes. The width of each pulse is determined by the relative importance of the growing and the damping effects. The equilibrium amplitude increases as the dispersive effect increases.

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Formations of a row of saturated solitonlike pulses in an unstable, dissipative, and dispersive nonlinear system have been observed in the numerical solutions of the following equation:

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \quad (1)$$

where α, β, and γ are positive constants characterizing instability (self-excitation), dispersion, and dissipation, respectively. Equation (1) can be used to describe the long waves on a viscous fluid flowing down an inclined plane[1] and the unstable drift waves in plasma (the dissipative trapped-ion modes with dispersion due to the finite ion banana width).[2] For $\beta = 0$, Eq. (1) reduces to that equation describing the chemical reactions which exhibit a turbulentlike behavior.[3]

Substituting $u \propto \exp(ikx + \sigma t)$ into the linearized version of (1), one obtains the linear dispersion relation $\sigma = \alpha k^2 - \gamma k^4 + i\beta k^3$. Thus small-amplitude sinusoidal waves are linearly unstable (growing) for long wavelengths and stable (damping) for short wavelengths. The maximum growth rate occurs at the wave number $k_{max} = (\alpha/2\gamma)^{1/2}$.

The existence of both instability and dispersion indicates the possibility of a steady state, because the energy influx due to the self-excitation is transferred through mode coupling to short wavelengths and is expected to be balanced by damping due to the fourth-derivative dissipation term. Energetically a steady state (in a statistical sense) is anticipated for $\beta = 0$, but the temporal evolution of the waveform has been found to exhibit a turbulentlike behavior.[3]

The purpose of this Letter is to investigate what role the dispersion ($\beta \neq 0$) plays in such an energy-transfer process. To do so, spatially periodic solutions of Eq. (1) for combinations of parameters α, β, and γ were obtained numerically by a finite-difference method in space and the Runge-Kutta-Gill method in time. 200 spatial mesh points were taken in the periodicity length $L = 2$ with periodic boundary conditions at $x = 0, L$. Initial conditions assigned were (a) Gaussian random variables[4] and (b) $\cos\pi x$. Among the numerical results obtained so far, those with fixed $\alpha = 1.0 \times 10^{-2}$ and $\gamma = 5.066 \times 10^{-6}$ are shown in Figs. 1–3.

Figure 1 shows the temporal evolution of u for $\beta = 0$ with initial condition (a). The wave components whose wavelengths correspond approximately to k_{max} soon predominate. Waves with roughly triangular shape appear and interact with

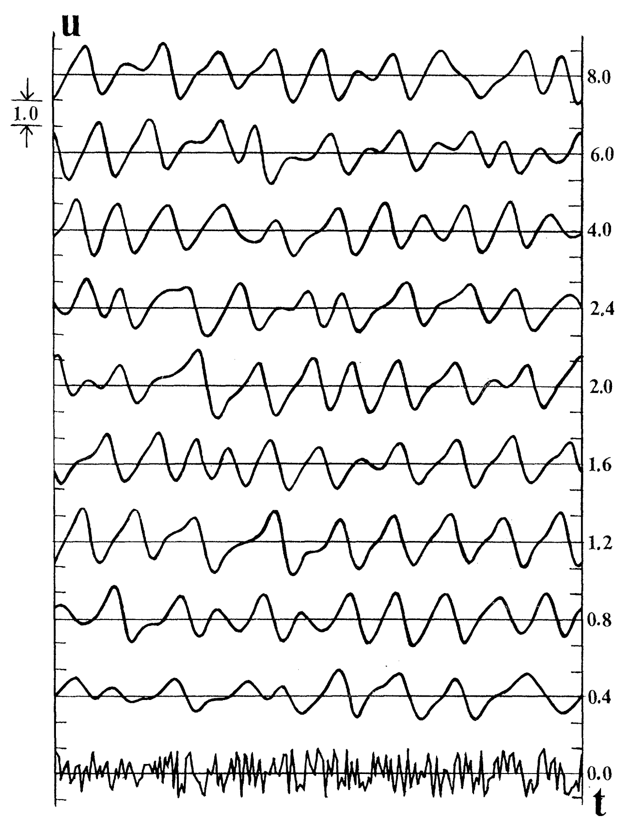

FIG. 1. Temporal evolution of u for initial condition (a) with $\alpha = 1.0 \times 10^{-2}$, $\beta = 0$, and $\gamma = 5.066 \times 10^{-6}$.

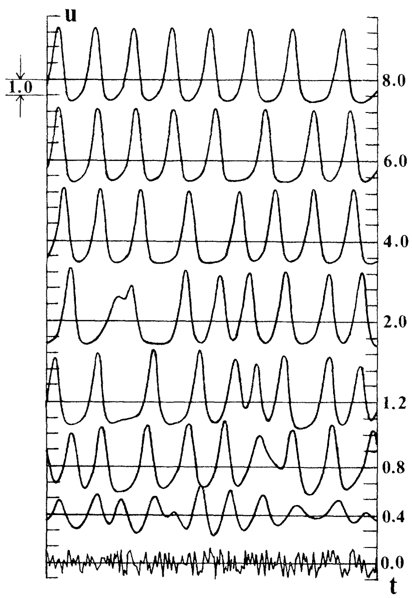

FIG. 2. Temporal evolution of u. The same as Fig. 1 except for $\beta = 4.84 \times 10^{-4}$.

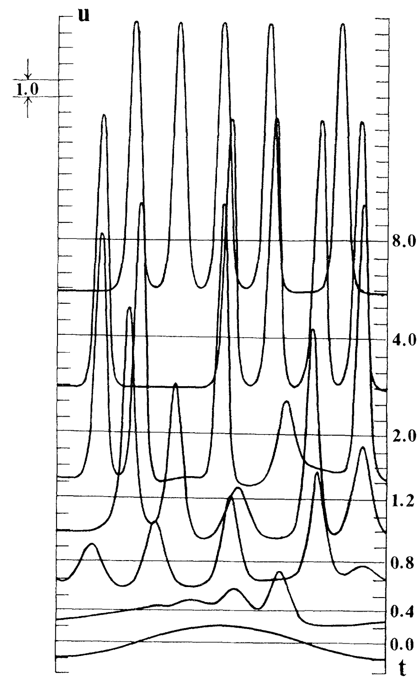

FIG. 3. Temporal evolution of u for initial condition (b) with $\beta = 2.0 \times 10^{-3}$.

each other without further growth in average amplitude. No regularity or no formation of organized structure has been observed up to $t = 8.0$. This is a reproduction of the turbulentlike behavior in chemical reactions.[3]

A temporal evolution is shown in Fig. 2 for $\beta = 4.84 \times 10^{-4}$ with other conditions kept the same as in Fig. 1. It is seen that waves corresponding to k_{max} are initially formed at around $t = 0.4$; they grow while interacting with each other and develop into a row of pulses of equal amplitude at $t = 4.0$. Once the amplitudes of the pulses are equalized, the row of pulses travels as a whole. It is interesting to note that the dispersion contributes to form a kind of organized structure.

In Fig. 3, the temporal evolution is shown for a strongly dispersive case $\beta = 2.0 \times 10^{-3}$ with initial condition (b). The initial smooth waveform $\cos \pi x$ develops several humps, similar to soliton breakup, after wave steepening by nonlinearity. The humps interact with each other while growing. At $t = 4.0$, the generated pulses almost line up into a row of solitonlike pulses of equal amplitude and travel as a whole up to $t = 8.0$.

Within the numerical computations carried out so far, the following general features have been found:

(i) The saturated amplitude of a row of pulses is constant irrespective of initial conditions for fixed α, β, and γ, but the number of pulses that emerges depends on initial conditions. For example, five pulses appear in Fig. 3 for the initial condition (b) but nine for (a) with other conditions kept unchanged.

(ii) The equilibrium amplitude increases and each pulse approaches the Korteweg–de Vries soliton when the dispersion becomes strong. As the relative importance of dispersion decreases, each equilibrium pulse deviates from a symmetric soliton shape and increases its asymmetry as is seen in Fig. 2.

(iii) The numerical result in Fig. 3 shows that the magnitudes of the terms in Eq. (1) are of comparable order initially, but those of the nonlinear and the dispersion terms become one order higher when the equilibrium state is attained. Normalization of (1) yields the parameter $\delta \equiv \beta/(\alpha\gamma)^{1/2}$ which represents the relative importance of the dispersion. Although the critical value of δ at which transition from a turbulentlike to an equilibrium state occurs has not been fixed yet, the equilibrium solitonlike pulses exist at least for $\delta \gtrsim O(1)$. For numerical computations in Figs. 2 and 3, δ is given by 2.15 and 8.89, respectively.

It is interesting to consider the case when $\beta u_{xxx} \sim uu_x \gg \alpha u_{xx} \sim \gamma u_{xxxx}$. If $\alpha = \gamma = 0$, Eq. (1) reduces to the Korteweg–de Vries equation and admits a soliton solution

$$u = N_0 + N\,\mathrm{sech}^2\{(N/12\beta)^{1/2}[x - (N_0 + \tfrac{1}{3}N)t]\}. \tag{2}$$

In connection with the above points, I consider the effect of finite but small α and γ, say $O(\epsilon)$, on solution (2). This problem can be treated by the two-timing asymptotic expansion.[5] Introducing the slow time scale $T \equiv \epsilon t$, we seek a solution of the form

$$u(x,t;\epsilon) = u^{(0)}(\eta,t,T) + \epsilon u^{(1)}(\eta,t,T) + O(\epsilon^2), \tag{3}$$

$$\eta \equiv [N(T)/12\beta]^{1/2}\{x - \int_0^t [N_0 + \tfrac{1}{3}N(T)]dt\}, \tag{4}$$

where η is a normalized new space coordinate in a moving frame and N is assumed to vary slowly in time. Substitution of (3) and (4) into (1) yields perturbation equations. The order-unity equation gives $u^{(0)} = N_0 + N(T)\mathrm{sech}^2\eta$. In order that (3) be valid for long times, as long as $t \sim O(\epsilon^{-1})$, we require nonsecular behavior of $u^{(1)}$ with respect to t. This condition yields a first-order differential equation for $N(T)$,

$$\frac{dN}{dT} = \frac{4\gamma}{189\beta^2}\left(\frac{21\alpha\beta}{5\gamma} - N\right)N^2. \tag{5}$$

It can be seen that the constant solution $N_\infty = 21\alpha\beta/5\gamma$ is stable and $N(T)$ approaches N_∞ monotonically.[6] For this value of N_∞, the characteristic width of the soliton is given by $l = (12\beta/N_\infty)^{1/2} = (20\gamma/7\alpha)^{1/2}$ and the propagation speed is $N_0 + \frac{1}{3}N_\infty = N_0 + 7\alpha\beta/5\gamma$.

For α, β, and γ used in Fig. 3, the theoretical estimate of the equilibrium amplitude is $N_\infty = 16.85$, whereas the numerically obtained amplitude is 16.64. The propagation velocity of a row of pulses after $t = 4.0$ in Fig. 3 is $N_0 + \frac{1}{3}N_\infty \simeq 2.41$, while the analytical result gives 2.37. The observed equilibrium amplitude 4.65 in Fig. 2 is larger than the theoretical estimate, 4.01, due to the soliton assumption. However, the asymmetry of the pulse shape in Fig. 2 indicates that a soliton is not a good first-order estimate because α and γ are important in this case.

Note that the characteristic width l is determined solely by α and γ and it is smaller than, but of comparable order to, the wavelength at the maximum growth rate, i.e., $l = (20\gamma/7\alpha)^{1/2} \simeq 0.19(2\pi/k_{\max})$. Furthermore, the amplitude of the soliton is proportional to β, so that the saturation amplitude increases as the dispersive effect increases.

A wave with sufficiently small amplitude and large width grows because the growth term αu_{xx} is more important than the damping term γu_{xxxx} for small wave numbers. Since the Korteweg-de Vries soliton decreases its width with increasing amplitude, the width of a growing wave decreases and increases the relative effect of damping leading to a slowdown of the rate of amplitude increase. Meanwhile, the dispersion can inhibit mode coupling and result in saturation at higher amplitudes for sufficient dispersion, which is responsible for the fact that N_∞ is proportional to β. Therefore, N asymptotically approaches the value N_∞ for which the growth just balances the damping and also the dispersion balances the nonlinearity.

In conclusion, I emphasize that computer solutions show the growth of an initial perturbation followed by formations of a row of solitons for the strongly dispersive case. Also to be emphasized is that the existence of a dispersive effect can bring about a kind of organization in the system that exhibits a turbulentlike behavior if the effect of dispersion is completely neglected. Further detailed numerical results will be reported elsewhere.

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