Kinetic Theory for Plasmas with Non-Abelian Interactions

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(Received 21 April 1983)

The derivation of kinetic equations for the classical and quantum (Wigner) distribution functions for a plasma with non-Abelian interactions is outlined. Particular emphasis is put on the relation between the color structures in the classical and the quantum theories.

PACS numbers: 12.35.Eq, 25.70.Np, 51.10.+y, 52.25.Dg

The recent widespread interest in using high-energy heavy-ion collisions as a means to create a quark-gluon plasma,[1] and thus to test the prediction of a transition from a color-confining to an unconfining (plasma) phase in QCD at high temperature and/or density,[1,2] has focused on a discussion of lepton-pair, photon, and strange-particle spectra,[1,3,4] which are supposed to contain signals for a transient plasma phase. Usually such computations are performed on the basis of the following assumptions: (i) The plasma phase forms instantaneously as soon as the critical value for the energy density (around 1 GeV/fm^3) is exceeded; (ii) local thermal and chemical equilibrium are established extremely fast and can be taken to persist at practically all times.[4,5]

Both assumptions are not yet tested. In fact, the expected lifetime for the plasma phase is short; numbers in the literature for, e.g., U + U collisions at c.m. energies of 20–50 GeV/nucleon vary between 4 and 8 fm/c.[3-6] Therefore, we suspect that, contrary to assumption (ii), the plasma phase spends a nonnegligible part of its lifetime trying to approach an equilibrium state. The existence of such a preequilibrium phase may change the predictions for measurable signatures of the plasma phase. To check this, a formalism for a dynamical, nonequilibrium description of the quark-gluon plamsa is required. As far as I know, this does not yet exist, and its formulation will be the subject of this Letter.

Before entering into an outline of the theory (a detailed account will be published elsewhere[7]), I mention another so far neglected feature of the quark-gluon plasma: Since color is no longer confined to hadrons in this phase, macroscopic (i.e., with a length scale $\gg 1$ fm) color fluctuations are possible. In order to be able to handle such effects, the kinetic theory will consistently treat all color degrees of freedom. Color fluctuations introduce the concept of "color conductivity,"[8] an important property of hadronic matter; its coefficient in the plasma phase will, in principle, be calculable within the present formalism.

The presentation will be in two steps: (1) Formulation of a *classical* kinetic theory and derivation of a classical chromohydrodynamics (designed for practical applications); (2) its justification through a *quantum-mechanical* treatment using relativistic Wigner functions.

Classical theory.—In a first step we try to describe the quark-gluon plasma as a system of classical colored particles and antiparticles (quarks and antiquarks) interacting with each other via a classical non-Abelian field $A_\mu{}^a(x)$.[9,10] We define a one-particle distribution function $f(x,p,Q)$ on "extended phase space" spanned by space-time coordinates x^μ, kinetic moment p^ν, and color vectors Q^a [$a = 1,\ldots,8$ for SU(3)]. Color has to be included into the definition of phase space[8]: Because of its ability to exchange color with the non-Abelian field under whose influence it is moving, the color charge of a classical colored particle, like its momentum, is a continuously varying function of time. Q_a rotates in color space, with Q^aQ_a and $d_{abc}Q^aQ^bQ^c$ being constants of motion. This is reflected in Wong's equations of motion[11] for a classical colored particle in a Yang-Mills (YM) field, which can be used[8] to derive a kinetic equation for the distribution function:

$$[p^\mu\partial_\mu + Q_aF_{\mu\nu}{}^a(x)p^\nu\partial_p{}^\mu + f_{abc}p^\nu A_\nu{}^b(x)Q^c\partial_Q{}^a]f(x,p,Q) = C[f,\bar{f}](x,p,Q), \qquad (1a)$$

$$[p^\mu\partial_\mu - Q_aF_{\mu\nu}{}^a(x)p^\nu\partial_p{}^\mu + f_{abc}p^\nu A_\nu{}^b(x)Q^c\partial_Q{}^a]\bar{f}(x,p,Q) = \bar{C}[f,\bar{f}](x,p,Q). \qquad (1b)$$

f and $\bar{f}$ are the distribution functions for quarks and antiquarks, respectively. C and $\bar{C}$ are collision terms, here for convenience chosen to be of the Boltzmann type[7] and hence depending only on f and $\bar{f}$ themselves.

Equations (1a) and (1b) turn into a genuine Vlasov-Boltzmann equation by requiring for the non-Abelian field the self-consistency condition

$$(D_\mu F^{\mu\nu})_a(x) = j_a{}^\nu(x), \tag{2}$$

with $j_a{}^\mu(x)$ from Eq. (4b), and $D_{ac}{}^\mu \equiv \partial^\mu \delta_{ac} - f_{abc} A_b{}^\mu(x)$.

From the kinetic equations (1) a set of macroscopic equations can be derived. From baryon-number conservation we find

$$\partial_\mu b^\mu(x) = 0;\; b^\mu(x) \equiv \int p^\mu (f - \bar{f}) dP\, dQ. \tag{3}$$

Here the integral goes over the momentum and color sectors of phase space with the measure

$$dP \equiv 2\theta(p_0)\delta(p^2 - m^2) d^4p,\;\; dQ \equiv \delta(Q^a Q_a - Q^2)\delta(d_{abc}Q^a Q^b Q^c - Q^3) d^8Q.$$

[d^8Q is the invariant group measure for the octet representation of SU(3).[7]] Color conservation yields

$$(D_\mu j^\mu)_a(x) = 0;\;\; j_a{}^\mu(x) \equiv \int p^\mu Q_a (f - \bar{f}) dP\, dQ. \tag{4}$$

Conservation of energy and momentum requires

$$\partial_\mu T_{\mathrm{MAT}}{}^{\mu\nu}(x) = -j_\mu{}^a(x) F_a{}^{\mu\nu}(x) \equiv -\partial_\mu T_{\mathrm{YM}}{}^{\mu\nu}(x), \tag{5a}$$

with

$$T_{\mathrm{MAT}}{}^{\mu\nu}(x) \equiv \int p^\mu p^\nu (f + \bar{f}) dP\, dQ \tag{5b}$$

being the energy-momentum tensor of the particles, and $T_{\mathrm{YM}}{}^{\mu\nu}$ being the one for the non-Abelian field.

The set of equations (2)–(5a) form the framework of a classical, relativistic, colored hydrodynamics[9] and are the relativistic generalization of the *chromohydrodynamics* recently proposed by Gibbons, Holm, and Kupershmidt.[12]

For later use, note that the Vlasov-Boltzmann equations (1) are equivalent to an *infinite* hierarchy of equations for the color moments of $\overset{(-)}{f}$, namely,

$$\overset{(-)}{f}(x,p) \equiv \int \overset{(-)}{f}(x,p,Q) dQ;\;\; \overset{(-)}{f}_a(x,p) \equiv \int Q_a \overset{(-)}{f}(x,p,Q) dQ,$$

etc. With the definition $g(x,p) \equiv f(x,p) + \bar{f}(x,-p)$, and similarly for the higher moments, and with the collision terms omitted, the hierarchy reads

$$p^\mu \partial_\mu g(x,p) = p^\mu F_{\mu\nu}{}^a(x) \partial_p{}^\nu g_a(x,p), \tag{6a}$$

$$p^\mu [\partial_\mu \delta_{ac} - f_{abc} A_\mu{}^b(x)] g_c(x,p) = p^\mu F_{\mu\nu}{}^b(x) \partial_p{}^\nu g_{ab}(x,p), \tag{6b}$$

$$p^\mu [\partial_\mu \delta_{ac}\delta_{bd} - f_{amc}\delta_{bd} A_\mu{}^m(x) - \delta_{ac} f_{bmd} A_\mu{}^m(x)] g_{cd}(x,p) = p^\mu F_{\mu\nu}{}^c(x) \partial_p{}^\nu g_{abc}(x,p), \tag{6c}$$

etc. This concludes the presentation of the classical kinetic theory in this Letter, and we now turn to a quantum-mechanical (QCD) treatment.

Quantum-mechanical treatment.—The quantum-mechanical analog of the one-particle distribution function is generated by the following operator:

$$\hat{\mathfrak{F}}(x,p) = (2\pi)^{-4} \int d^4v\, e^{-ip\cdot v} : \hat{\bar{\Psi}}(x + \tfrac{1}{2}v) P\{\exp[-i\tilde{Q}_a \int_x^{x+v/2} \hat{A}_\mu{}^a(z) dz^\mu]\} \otimes P\{\exp[-i\tilde{Q}_a \int_{x-v/2}^{x} \hat{A}_\mu{}^a(z) dz^\mu]\} \hat{\Psi}(x - \tfrac{1}{2}v):, \tag{7}$$

where $\tilde{Q}_a = -\lambda_a/2$, and $\hat{\Psi}$ solves the Dirac equation $i\gamma^\mu[\partial_\mu + i\tilde{Q}_a \hat{A}_\mu{}^a(x)]\hat{\Psi}(x) = m\hat{\Psi}(x)$. $\hat{\mathfrak{F}}$ is a matrix operator with color indices $A,B = 1,2,3$ and spinor indices α,β. The path-ordered exponentials make physical quantities (to be calculated as ensemble expectation values of gauge-covariant operators with $\hat{\mathfrak{F}}$) *gauge covariant;* hence, (7) is the gauge-invariant generalization of the Wigner operator usually found in textbooks.[13] $\langle \hat{\mathfrak{F}} \rangle$ from (7) is as close as one can get to a positive-definite distribution function in quantum mechanics.[13] The macroscopic observables $b_\mu(x)$, $j_\mu{}^a(x)$, and $T_{\mu\nu}(x)$, expressed through $\hat{\mathfrak{F}}$, take the following form:

$$b^\mu(x) \equiv \langle : \hat{\bar{\Psi}}(x) \gamma^\mu \hat{\Psi}(x) : \rangle = \int d^4p\, \langle \mathrm{Tr}\, \gamma^\mu \hat{\mathfrak{F}}(x,p) \rangle = \int d^4p\, p^\mu \langle \mathrm{Tr}\, \hat{\mathfrak{F}}(x,p) \rangle + \text{spin effects}, \tag{8}$$

$$j_a{}^\mu(x) \equiv \langle : \hat{\bar{\Psi}}(x) \gamma^\mu \tilde{Q}_a \hat{\Psi}(x) : \rangle = \int d^4p\, \langle \mathrm{Tr}\, \gamma^\mu \tilde{Q}_a \hat{\mathfrak{F}}(x,p) \rangle = \int d^4p\, p^\mu \langle \mathrm{Tr}\, \tilde{Q}_a \hat{\mathfrak{F}}(x,p) \rangle + \text{spin effects}, \tag{9}$$

$$T_{\mathrm{MAT}}{}^{\mu\nu}(x) \equiv (i/2) \langle : \hat{\bar{\Psi}}(x) \gamma^\mu \vec{\mathrm{D}}^\nu \hat{\Psi}(x) - \hat{\bar{\Psi}}(x) \overleftarrow{\mathrm{D}}^{+\nu} \gamma^\mu \hat{\Psi}(x) : \rangle = \int d^4p\, p^\mu \langle \mathrm{Tr}\, \gamma^\nu \hat{\mathfrak{F}}(x,p) \rangle = \int d^4p\, p^\mu p^\nu \langle \mathrm{Tr}\, \hat{\mathfrak{F}}(x,p) \rangle + \text{spin effects}. \tag{10}$$

In these equations, spin effects have been isolated through a Gordon decomposition as terms involving $\sigma_{\mu\nu}$, and are henceforth neglected.[14] The quantities (8)–(10) obey the chromohydrodynamical equations (3)–(5a). We see that everything comes out analogous to the classical theory, up to the facts that the momentum integrals are not on mass shell, and that the color integrals are replaced by traces over color indices. We will now focus on an understanding of these differences.

To this end we split $\hat{\mathfrak{F}}$ in the following way:

$$\hat{\mathfrak{F}}(x,p)=\hat{\mathfrak{F}}^{(+)}(x,p)+\hat{\mathfrak{F}}^{(-)}(x,p)+\hat{\mathfrak{F}}^{(Z)}(x,p);\quad \hat{\mathfrak{F}}^{(\pm)}(x,p)\equiv\theta(\pm p_0)\theta(p^2)\hat{\mathfrak{F}}(x,p),\quad \hat{\mathfrak{F}}^{(Z)}(x,p)\equiv\theta(-p^2)\hat{\mathfrak{F}}(x,p).$$

The last term (attributed to the *Zitterbewegung*[13]) is of no importance classically.[13] $\hat{\mathfrak{F}}^{(+)}$ and $\hat{\mathfrak{F}}^{(-)}$ usually are peaked near the mass shell[13] and classically correspond to $\theta(p_0)\delta(p^2-m^2)f(x,p,Q)$ and $\theta(-p_0)\delta(p^2-m^2)\bar{f}(x,-p,Q)$, respectively. We expect this correspondence to become explicit in a semiclassical ($\hbar\to 0$) approximation.

Next we discuss the color structure of $\hat{\mathfrak{F}}(x,p)$. (We here neglect the spin indices of $\hat{\mathfrak{F}}$.) The color matrix $\hat{\mathfrak{F}}(x,p)$ may be expanded as

$$\langle\hat{\mathfrak{F}}_{AB}\rangle=\tfrac{1}{3}\delta_{AB}\underline{g}+2\sum_{a=1}^{8}(\tilde{Q}_a)_{AB}\underline{g}_a, \tag{11a}$$

where

$$\underline{g}\equiv\mathrm{Tr}_C\langle\hat{\mathfrak{F}}\rangle;\quad \underline{g}_a\equiv\mathrm{Tr}_C\langle\tilde{Q}_a\hat{\mathfrak{F}}\rangle. \tag{11b}$$

Insertion into Eqs. (8)–(10) and comparison with (3)–(5) reveals the following correspondence between the moments (11b) of the Wigner function and their classical counterparts:

$$\underline{g}(x,p)\to\delta(p^2-m^2)\{\theta(p_0)\int f(x,p,Q)\,dQ+\theta(-p_0)\int\bar{f}(x,-p,Q)\,dQ\};$$

$$\underline{g}_a(x,p)\to\delta(p^2-m^2)\{\theta(p_0)\int Q_a f(x,p,Q)\,dQ+\theta(-p_0)\int Q_a\bar{f}(x,-p,Q)\,dQ\}.$$

Thus, although $\underline{g}(x,p)$ and $\underline{g}_a(x,p)$ completely specify the Wigner function, their knowledge allows us to determine only the two lowest color moments of the Q-dependent classical distribution function. The reason is that quantum mechanically the higher moments [i.e., $\langle\mathrm{Tr}(\tilde{Q}_a\tilde{Q}_b\hat{\mathfrak{F}})\rangle$, etc.] are related to $\underline{g}$ and $\underline{g}_a$ via the SU(3) algebra, whereas no such relations exist *a priori* for the higher color moments of the classical distribution function. Imposing them in the classical case would enforce the color algebra also for the classical theory. This, however, is not in the spirit of the classical approach: Together with the Vlasov equation we would simultaneously have to solve an infinite number of constraint equations, thus losing the simplicity of the classical formalism. The philosophy that I adopt instead will become clear below.

We now derive a transport equation for the Wigner function in QCD.[15] To reproduce the Vlasov limit (incorporating the effect of a self-consistent non-Abelian field) we assume that the YM field develops a nonvanishing expectation value: $\hat{A}_\mu{}^a(x)=A_\mu{}^a(x)+\delta\hat{A}_\mu{}^a(x)$. The equation for $\langle\hat{\mathfrak{F}}\rangle$ can be split up as follows:

$$p^\mu\partial_\mu\underline{g}(x,p)=p^\mu F_{\mu\nu}{}^a(x,p)\partial_p{}^\nu\underline{g}_a(x,p)+\text{collision terms}, \tag{12a}$$

$$p^\mu[\partial_\mu\delta_{ac}-f_{amc}A_\mu{}^m(x)]\underline{g}_c(x,p)=p^\mu F_{\mu\nu}{}^b\partial_p{}^\nu\underline{g}_{(ab)}(x,p)+\text{collision terms}. \tag{12b}$$

Here the brackets in $\underline{g}_{(ab)}(x,p)$ mean symmetrization: $\underline{g}_{(ab)}\equiv\frac{1}{2}\langle\mathrm{Tr}\{\tilde{Q}_a,\tilde{Q}_b\}\hat{\mathfrak{F}}\rangle$. The color algebra can be used to express $\underline{g}_{(ab)}$ in terms of $\underline{g}$ and $\underline{g}_a$, and the system (12) closes.

The collision terms in (12) contain correlations involving the quantum field $\delta\hat{A}_\mu{}^a(x)$ and higher momentum derivatives of $\hat{\mathfrak{F}}(x,p)$ (Ref. 7) [the latter again vanish in the limit $\hbar\to 0$ (Ref. 16)]. They are presently being worked out; once evaluated, they will allow the computation of transport coefficients for the plasma,[7] leading, hopefully, to a more intuitive understanding of its physical properties.

Instead of using the color algebra to close the system (12), one may derive transport equations for the higher moments too. One finds

$$p^\mu[\partial_\mu\delta_{ac}\delta_{bd}-f_{amc}\delta_{bd}A_\mu{}^m(x)-\delta_{ac}f_{bmd}A_\mu{}^m(x)]\underline{g}_{(cd)}(x,p)$$
$$=p^\mu F_{\mu\nu}{}^c(x)\partial_p{}^\nu\underline{g}_{((ab)c)}(x,p)+\text{collision terms}, \tag{12c}$$

etc. Up to the only partial symmetrization of color indices, this hierarchy is formally identical to the classical analog (6). Whereas in the quantum case the hierarchy is generated by the color algebra (through construction of higher moments from lower ones), in the classical case it is generated by the non-Abelian term in (1). This demonstrates the importance of this term and shows that the Vlasov equation (1) is the "correct" (or at least most closely related to the quantum theory) classical transport equation for a non-Abelian plasma.

The analogy between the classical and the quantum transport equations for a non-Abelian plasma demonstrated here allows for an implementation of quantum-mechanically calculated collision terms in the classical kinetic theory. Furthermore, it provides a justification for applying the chromohydrodynamics (1)–(5) obtained in this way to the quark-gluon plasma in heavy-ion collisions.

I thank A. Chodos and V. Moncrief for fruitful discussions which stimulated this work. I also gratefully acknowledge helpful remarks from B. Müller, A. Schäfer, and W. Greiner. This work was supported by the Deutsche Forschungsgemeinschaft.

[1]For a review see *Quark Matter Formation and Heavy Ion Collisions*, Proceedings of the Bielefeld Workshop, May 1982, edited by M. Jacob and H. Satz (World Scientific, Singapore, 1982).

[2]For a review see E. Shuryak, Phys. Rep. 61C, 71 (1980); D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[3]E. L. Feinberg, Nuovo Cimento 34A, 391 (1976); G. Domokos and J. Goldman, Phys. Rev. D 23, 203 (1981); K. Kajantie and H. I. Miettinen, Z. Phys. C 9, 341 (1981), and 14, 357 (1982); P. Koch, J. Rafelski, and W. Greiner, Phys. Lett. 123B, 151 (1983); P. Koch, diploma thesis, Johann Wolfgang Goethe University, 1983 (unpublished).

[4]J. Rafelski and B. Müller, Phys. Rev. Lett. 48, 1066 (1982).

[5]R. Anishetty, P. Koehler, and L.McLerran, Phys. Rev. D 22, 2793 (1980).

[6]J. D. Bjorken, Phys. Rev. D 27, 140 (1983).

[7]U. Heinz, to be published.

[8]G. Baym, Physica (Utrecht) 96A, 131 (1979).

[9]U. Heinz, in Ref. 1, p. 439; U. Heinz, A. Chodos, and V. Moncrief, "A Relativistic Vlasov-Equation for Plasmas with Non-Abelian Interactions" (unpublished).

[10]Here spin and flavor degrees of freedom are neglected. Inclusion of flavor leads to a many-fluid hydrodynamics. Spin can be included analogously to color, classically with use of Arodź's [H. Arodź, Phys. Lett. 116B, 251 (1982)] instead of Wong's equations [S. K. Wong, Nuovo Cimento 65A, 689 (1970)], and quantum mechanically by expansion of the Wigner function also into Dirac matrices [see Eq. (11)].

[11]Wong, Ref. 10.

[12]J. Gibbons, D. D. Holm, and B. Kupershmidt, Phys. Lett. 90A, 281 (1982).

[13]S. R. deGroot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).

[14]We realize, however, the importance of spin-color couplings (see Arodź, Ref. 10) and chromomagnetic effects in QCD, requiring a thorough treatment of these terms in future work.

[15]A similar equation for a scalar theory was derived by S.-P. Li and L. McLerran, Nucl. Phys. B214, 417 (1983).

[16]E. Wigner, Phys. Rev. 40, 749 (1932); see also P. Carruthers and F. Zachariasen, Rev. Mod.Phys. 55, 254 (1983).