Are Three-Frequency Quasiperiodic Orbits to Be Expected in Typical Nonlinear Dynamical Systems?

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The current state of theoretical understanding related to the question posed in the title is incomplete. This paper presents results of numerical experiments which are consistent with a positive answer. These results also bear on the problem of characterizing possible *routes to chaos* in nonlinear dynamical systems.

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Landau proposed, as a possible mechanism for the onset of turbulence, the successive destabilization of fluid modes of incommensurate frequency. According to this picture, as a stress parameter of the system (e.g., the Reynolds number) is increased, successive discrete frequencies appear in the Fourier power spectrum of the fluid variables along with their integer harmonic sum and difference combinations. The system time dependence then appears to become very complex (turbulent) when many frequencies are present. This view was challenged in the paper by Ruelle and Takens,[1] who proposed that truly chaotic time dependence can result after only a few bifurcations. Furthermore, they offered a specific mechanism by which this might occur. The related subsequent paper of Newhouse, Ruelle, and Takens[2] showed that, if one has a system with a phase-space "flow" consisting of three incommensurate frequencies, then there exist arbitrarily small changes of the system which convert the flow from a quasiperiodic, three-frequency flow to one which is chaotic.[3] One might naively conclude, on this basis, that three-frequency flow was unlikely, since it can be destroyed by small perturbations. On the other hand, the small perturbations necessary to make the flow chaotic may have to be very delicately chosen and hence may be unlikely to occur in practice. The mathematical proofs of the existence of these perturbations[1,2] do not give us a satisfactory answer to this problem. In fact, in the proof of Ref. 2, the small perturbations necessary to create chaotic attractors have small first and second derivatives but may not necessarily also have small third and higher-order derivatives. For physical applications, however, we expect that smooth perturbations (which do not have large third and higher-order derivatives) are of most relevance (it is only such perturbations that we shall be concerned with in this paper). Thus, theoretical understanding related to the question of whether one should typically expect to observe three-frequency quasiperiodicity in experiments is currently very incomplete. In the present paper we attempt a series of numerical experiments with a view toward making some progress in clarifying this situation. We find that, for the definite system studied, three-frequency quasiperiodicity can occur and is, in fact, fairly common. This is analogous to the rigorous result of Arnol'd, who showed that two-frequency quasiperiodicity is stable to small changes in the same (measure theoretic) sense as our numerical results suggest three-frequency quasiperiodicity to be.[4]

These results also bear on the subject of "routes to chaos" in typical dynamical systems. By a route to chaos we mean an answer to the following question: Given a dynamical system with only nonchaotic, stable, time-asymptotic dynamical states, how do chaotic attractors arise as some parameter of the system is varied? Several ways (routes) by which this can occur have now been well documented. These include infinite period-doubling cascades,[5] intermittency,[6] and crises.[7] The existence of the second and third routes confirms, in a general way, Ruelle and Takens's proposal that chaos can arise after only a few bifurcations. However, the specific mechanism by which they speculated this might happen[1] remains in doubt. In particular, on the basis of Refs. 1 and 2, another route to chaos is also sometimes thought to be possible.[1,8] Namely, if a flow arises with two-frequency quasiperiodicity, the destabilization of a third incommensurate frequency is supposed instantly to create a chaotic flow. Our results suggest that this scenario is unlikely in typical physical situations.[9]

Saying that $x(t)$ is quasiperiodic with three frequencies, f_1, f_2, and f_3 means that $x(t)$ has the form $x(t)=F(f_1t, f_2t, f_3t)$, where $F(u, v, w)$ is periodic with period 1 in (u, v, w). [Expressing

F as a Fourier series in (u, v, w), we then see that the frequency power spectrum of $x(t)$ is composed entirely of discrete spectral components at frequencies $pf_1+qf_2+rf_3$ where p, q, and r are integers.] Choice of a Poincaré surface of section so that the process is examined at times $t=n/f_1$, with n an integer, yields $x(n/f_1) = F(0, n\omega_1, n\omega_2)$, where $\omega_1=f_2/f_1$ and $\omega_2=f_3/f_1$. The physical system will actually have many dependent variables, $x_1(t)$, $x_2(t), \ldots$, all with the same type of three-frequency quasiperiodic dependence, $x_i(n/f_1)=F_i(0, n\omega_1, n\omega_2)$. The trajectories hit the Poincaré surface of section only at points of the form $(x_1, x_2, \ldots) = (F_1(0, \theta, \varphi), F_2(0, \theta, \varphi), \ldots)$ where $0 \leq \theta \leq 1$ and $0 \leq \varphi \leq 1$. Thus the attractor in the surface of section is a two-dimensional torus coordinatized by θ and φ. For successive returns to the surface of section, θ and φ change by the rule

$$\theta_{n+1}=[\theta_n+\omega_1]\,\mathrm{mod}1, \qquad (1a)$$

$$\varphi_{n+1}=[\varphi_n+\omega_2]\,\mathrm{mod}1. \qquad (1b)$$

The flow is truly three frequency if ω_1, ω_2, and 1 are incommensurate numbers [i.e., integers (p, q, r) do not exist for which $p\omega_1+q\omega_2+r=0$]. In this case Eqs. (1) generate an orbit which is ergodic on the θ-φ torus. Now we perturb Eqs. (1) as follows:

$$\theta_{n+1}=[\theta_n+\omega_1+\epsilon P_1(\theta_n, \varphi_n)]\,\mathrm{mod}1, \qquad (2a)$$

$$\varphi_{n+1}=[\varphi_n+\omega_2+\epsilon P_2(\theta_n, \varphi_n)]\,\mathrm{mod}1, \qquad (2b)$$

where the $P_{1,2}$ are periodic in θ and φ (since the map is on a torus), and we take $P_{1,2}\sim O(1)$. If Eqs. (2) have three-frequency quasiperiodic orbits, then, by definition, there exists a nonlinear change of variables, $(\theta, \varphi)\rightarrow(\theta', \varphi')$, such that the new variables satisfy Eqs. (1) with ω_1 and ω_2 replaced by ρ_1 and ρ_2, where ρ_1 and ρ_2 are the *winding numbers* generated by (2); viz., $\rho_1 \equiv \lim_{n\rightarrow\infty}(\theta_n-\theta_0)/n$; $\rho_2 \equiv \lim_{n\rightarrow\infty}(\varphi_n-\varphi_0)/n$; and for the purposes of these definitions of $\rho_{1,2}$ the mod1 should be deleted from (2) in calculating θ_n and φ_n. On the other hand, according to the theorem of Ref. 2, there exist arbitrarily small (not necessarily smooth) perturbations, $(\epsilon P_1, \epsilon P_2)$, such that the attracting orbits generated by (2) are chaotic. Whether this occurs for typical smooth $P_{1,2}$ is the question we wish to address. We find, in fact, that it does not occur for typical smooth $P_{1,2}$, and, furthermore, it does not even necessarily occur for large perturbations [i.e., $\epsilon P_{1,2}\sim O(1)$].

Making use of the periodicity of $P_{1,2}$ in (θ, φ), we express it as a Fourier sum of terms $A_{rs}\sin[2\times\pi(r\theta+s\varphi+B_{rs})]$, with integer (r, s) the summation indices. We pick (somewhat arbitrarily) a convenient form of $P_{1,2}$ by retaining only $(r, s) = (0,1)$, $(1,0)$, $(1,1)$, and $(1, -1)$. We then choose the amplitudes and phases, A_{rs} and B_{rs}, using a random-number generator. Thus, in some sense, we are picking the $P_{1,2}$ at random. All the results reported here are for one particular such choice. We emphasize, however, that other choices using other (r, s) combinations have been found to give similar results. For our coefficient choice we form J, the Jacobian determinant of the map, Eqs. (2). Note that this determinant depends on ϵ, θ, and φ but not on $\omega_{1,2}$. The map (2) will be invertible if $J\neq 0$ everywhere in $0 \leq (\theta, \varphi) \leq 1$. For the A and B coefficients chosen, this gives $\epsilon<\epsilon_c$, where $\epsilon_c \cong 0.673$. For $\epsilon>\epsilon_c$ the map is noninvertible. Three-frequency quasiperiodicity is possible only if $\epsilon \leq \epsilon_c$, since otherwise a transformation of variables to Eqs. (1) (which are invertible) would be impossible.

There are four possible types of attractors for the map Eqs. (2). These are (i) three-frequency quasiperiodic, (ii) two-frequency quasiperiodic, (iii) periodic (one frequency), and (iv) chaotic. We have found that, for our purposes, the most convenient method of numerically distinguishing between these four classes is by calculating their Lyapunov exponents, which we denote by h_1 and h_2, with the convention $h_1 \geq h_2$. For a three-frequency quasiperiodic orbit, $h_1=h_2=0$, since then (2) may be transformed to (1) for which the Jacobian matrix is the identity matrix. For two-frequency quasiperiodic attractors, ρ_1, ρ_2, and 1 are commensurate, the attractor is a closed curve winding around the (θ, φ) torus, and the Lyapunov exponents satisfy $h_1=0$, $h_2<0$. In this case, $h_2<0$ corresponds to the fact that the closed curve is attracting, while $h_1=0$ corresponds to the fact that motion along this curve is two-frequency quasiperiodic [i.e., a proper choice of variables can put the governing equation for motion along the attracting curve in the form of Eq. (2a)]. For an attracting N-periodic orbit, $(\theta_1, \varphi_1) \rightarrow (\theta_2, \varphi_2) \rightarrow \cdots \rightarrow (\theta_N, \varphi_N) \rightarrow (\theta_1, \varphi_1) \rightarrow \cdots$, ρ_1 and ρ_2 are both rational, and the Lyapunov exponents are both negative, $h_{1,2}<0$. For a chaotic attractor $h_1>0$. This condition may be taken as a definition of chaos, since it implies exponential growth of the separation between nearby orbit trajectories and hence sensitive dependence on initial conditions.

The Lyapunov-exponent characteristics of the

TABLE I. Lyapunov-number characterization of attractors and numerical test criteria.

Type of attractor	Lyapunov characterization	Numerical test criteria
Three-frequency quasiperiodic	$h_1=h_2=0$	$(\bar{h}_1^2+\bar{h}_2^2)^{1/2}<10^{-4}$
Two-frequency quasiperiodic	$h_1=0,\ h_2<0$	$(\bar{h}_1^2+\bar{h}_2^2)^{1/2}>10^{-4}$ and $\lvert\bar{h}_1\rvert<10^{-4}$
Periodic	$h_1<0,\ h_2<0$	$\bar{h}_1<-10^{-4}$
Chaotic	$h_1>0$	$\bar{h}_1>+10^{-4}$

TABLE II. Frequency of observation of different types of attractor for 256 random choices of (ω_1, ω_2) for each of three values of ϵ/ϵ_c.

Type of attractor	$\epsilon/\epsilon_c=\frac{3}{8}$	$\epsilon/\epsilon_c=\frac{3}{4}$	$\epsilon/\epsilon_c=\frac{9}{8}$
Three-frequency quasiperiodic	82%	44%	0%
Two-frequency quasiperiodic	16%	38%	33%
Periodic	2%	11%	31%
Chaotic	0%	7%	36%

various types of attractor discussed above are summarized in the second column of Table I. On the basis of these properties we have devised a test which we apply to Eqs. (2) with our random choice of the coefficients. Table II shows a summary of one set of results for three different values of ϵ, namely, $\epsilon/\epsilon_c=\frac{3}{8}$, $\frac{3}{4}$, and $\frac{9}{8}$. These results were obtained as follows. For each value of ϵ, 256 pairs of ω_1 and ω_2 were each chosen randomly with uniform distribution in the interval from 0 to 1. For each (ω_1, ω_2) the map was iterated 10^5 times and h_1 and h_2 calculated. (The same initial condition for θ and φ was used for all cases.) Note that, since the number of iterates is finite, the calculated Lyapunov exponents are only approximations to the true Lyapunov exponents. Let $\bar{h}_1$ and $\bar{h}_2$ denote our calculated estimates of h_1 and h_2. If $(\bar{h}_1^2+\bar{h}_2^2)^{1/2}<10^{-4}$, we judge the orbit to be three-frequency quasiperiodic; if $(\bar{h}_1^2+\bar{h}_2^2)>10^{-4}$ but $\lvert\bar{h}_1\rvert<10^{-4}$, we judge the orbit to be two-frequency quasiperiodic; if $\bar{h}_2\leqslant\bar{h}_1\leqslant-10^{-4}$, we judge the orbit to be periodic; and if $\bar{h}_1\geqslant10^{-4}$, we judge the orbit to the chaotic. These criteria are summarized in the third column of Table I. Table II shows the fraction of the orbits of each class for each value of ϵ. Thus, three-frequency quasiperiodic orbits were judged to occur for 82% of the choices of (ω_1, ω_2) for the case $\epsilon/\epsilon_c=\frac{3}{8}$, for 44% of the choices for $\epsilon/\epsilon_c=\frac{3}{4}$, and for none of the choices for $\epsilon/\epsilon_c=\frac{9}{8}$. (As previously mentioned, it is known on theoretical grounds that three-frequency quasiperiodicity cannot occur for $\epsilon>\epsilon_c$.) Furthermore, we note that after three-frequency quasiperiodicity, the most common occurrence for $\epsilon/\epsilon_c=\frac{3}{8}$ and $\frac{3}{4}$ is two-frequency quasiperiodicity, with the occurrence of periodic, and especially chaotic, orbits being comparatively rare in these cases. As a check to verify that $(\bar{h}_1^2+\bar{h}_2^2)^{1/2}<10^{-4}$ is a reasonable criterion for three-frequency quasiperiodicity, Fig. 1 shows a logarithmic plot of $(\bar{h}_1^2+\bar{h}_2^2)^{1/2}$ vs N, where N is the number of iterates used in calculating $\bar{h}_1$ and $\bar{h}_2$, for a case judged to be three-frequency quasiperiodic. The decrease in the envelope maximum of $(\bar{h}_1^2+\bar{h}_2^2)^{1/2}$ as N^{-1} is evident over four decades of variation.

The results reported here suggest that if arbitrarily small smooth perturbations exist which destroy three-frequency quasiperiodicity, then they must have to be very delicately chosen and are thus unlikely to occur in practice. In particular, we believe that for a fixed typical choice of P_1 and P_2, the measure of (ω_1, ω_2) which yields chaos approaches zero as $\epsilon\to0$. In conclusion, we have shown that the addition of smooth nonlinear perturbations does not typically cause the occurrence of three-frequency quasiperiodicity to cease. Furthermore, three-frequency quasi-

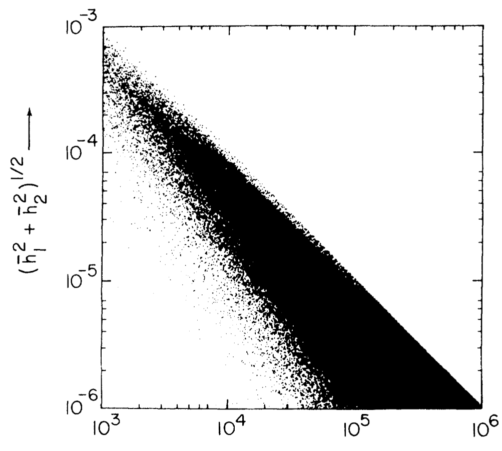

FIG. 1. Plot of $(\bar{h}_1^2+\bar{h}_2^2)^{1/2}$ vs the number of iterations (N) on a logarithmic scale for one of the cases in Table II which were judged to be three-frequency quasiperiodic. $\epsilon/\epsilon_c=\frac{3}{4}$, $\omega_1=0.424\,544\,965\,176\,41$, $\omega_2=0.126\,984\,126\,984\,13$.

periodicity persists and is prevalent even when large (e.g., $\epsilon/\epsilon_c = \frac{3}{4}$) nonlinear terms are introduced.[10,11]

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[11]In a future publication we shall provide further elaborations, discussions, and extensions of the results reported herein. In addition, the character of the chaotic attractors and how they arise as parameters are varied will be discussed, as well as extension to four-frequency flows.