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Inverse Scattering of First-Order Systems in the Plane Related to Nonlinear Multidimensional Equations

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A method for solving the inverse problem of certain hyperbolic as well as elliptic systems of n equations in the plane is given. This result can be used to linearize the initial-value problem of several physically significant nonlinear evolution equations in two spatial and one temporal dimension.

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It is well known[1] that the inverse problem of the Schrödinger eigenvalue equation has been used to linearize the initial-value problem of the Korteweg–de Vries (KdV) equation.[2] A similar role has been played by the so-called Ablowitz-Kaup-Newell-Segur (AKNS)[3] eigenvalue problem (a system of two equations) in connection with several physically important equations, e.g., the sine-Gordon, the nonlinear Schrödinger, the modified KdV, etc. The generalization of the AKNS problem to systems of n equations, which we call $n\times n$ AKNS, is also related to physical equations, e.g., the n-wave interaction.[4] The inverse problem of the $n\times n$ AKNS has been recently solved by Beals and Coifman.[5]

It has been further established[4,6] that certain two-spatial-dimensional analogs of the above eigenvalue problems are also related to physically interesting evolution equations. These equations are $(2+1)$-dimension, i.e., two space and one time, analogs of the nonlinear equations mentioned above. However, the question of finding a suitable method, such as the inverse scattering transform (IST), for solving the initial-value problem of these and other equations in $2+1$ dimensions remained open for a rather long time. In this regard I mention that some interesting results had been obtained in connection with the Kadomtsev-Petviashvili (KP) equation[7] (a $2+1$ analog of the KdV equation) and with the three-wave interaction in $2+1$ dimensions.[8] However, it was not clear from this work how an IST formalism for problems in $2+1$ dimensions could be developed.

The Schrödinger eigenvalue equation has been generalized to two dimensions in two different forms related to KP I and KP II (both of these equations arise naturally in various physical contexts[1]). The inverse problem corresponding to KP I has recently been linearized[9] with use of a Riemann-Hilbert boundary-value problem. Similarly a so-called "$\bar{\partial}$" problem has been used to linearize KP II.[10]

In this Letter I present a method for solving the inverse problem of a rather general system of n equations in the plane. Furthermore I show how this result can be used to linearize the initial-value problem of several physically important equations. In particular in sections (a) and (b) of this Letter I consider the two-dimensional generlization of the $n\times n$ AKNS problem proposed in Ref. 4:

$$\psi_x = \lambda B\psi + q\psi + J\psi_y. \tag{1}$$

In Eq. (1) B and J are $n\times n$ constant diagonal matrices and $q(x,y)$ is an $n\times n$ off-diagonal matrix containing the potentials (or field variables). I assume that $q(x,y)\to 0$ sufficiently fast for large x,y. I investigate both the hyperbolic (i.e., J real) as well as the elliptic (i.e., J imaginary) versions of (1). The hyperbolic and elliptic cases are linearized with the aid of a Riemann-Hilbert and a "$\bar\partial$" problem, respectively. In the elliptic case I assume that a certain linear integral equation characterizing suitable Jost eigenfunctions has no homogeneous solutions. If such homogeneous solutions exist they give rise to lumps, i.e., decaying solitons in 2 + 1 dimensions (the situation is similar to that found in Benjamin-Ono[11] and KP I[9]). In the latter case the formalism presented here must be appropriately modified to incorporate these homogeneous modes. This, as well as various rigorous aspects of this work, will be presented in the future[12] (for the hyperbolic case we do not expect lumps). In section (c) I indicate how the results of sections (a) and (b) can be used to solve the initial-value problem of certain nonlinear equations. Concrete results are given for the n-wave interaction in 2 +1 dimensions and for variants of the Davey-Stewartson (DS) equation.[13] The DS equation is a 2 +1 generalization of the nonlinear Schrödinger equation and arises generically in physical contexts (its various forms, for the case of water waves, are related to the existence or nonexistence of surface tension[1]). The modified KP equation, which is also contained in (1), can be treated in an exactly similar way.

(a) I first consider the inverse problem associated with the hyperbolic system:

$$\mu_x = ik\hat{J}\mu + q\mu + J\mu_y; \quad \hat{J}f \equiv Jf - fJ, \tag{2}$$

where μ is an nth-order matrix, J is a constant real diagonal matrix with elements $J_1 > J_2 \ldots > J_n$, and $q(x,y)$ is an nth-order off-diagonal matrix containing the potentials $q_{ij}(x,y)$. I assume that $q_{ij}(x,y)\to 0$ rapidly enough for large x, y. Equation (2) is obtained from the well-known[4] equation (1) by taking $B=0$ (this is without loss of generality) and using $\psi = \mu\exp[ik(Jx+y)]$. Equation (2) with $\partial/\partial y = 0$ has been investigated in Ref. 5. Let $\pi_0\mu$, $\pi_+\mu$, and $\pi_-\mu$ denote the diagonal, strictly upper diagonal, and strictly lower diagonal parts of the matrix μ. A solution of (2), bounded for all values of $k = k_R + ik_I$ and tending to the unit matrix I as $k\to\infty$, is given by $\mu(x,y,k) = \mu^+(x,y,k)$ for $k_I \geq 0$, $\mu(x,y,k)=\mu^-(x,y,k)$ for $k_I \leq 0$, where $\mu^\pm(x,y,k)$ satisfy the following linear integral equations:

$$\mu^\pm(x,y,k) = I + (1/2\pi)\int_{-\infty}^{x} d\xi\, E e^{ik(x-\xi)\hat{J}}(\pi_0+\pi_\pm)q(\xi,\eta)\mu^\pm(\xi,\eta,k) - (1/2\pi)\int_{x}^{\infty} d\xi\, E e^{ik(x-\xi)\hat{J}}\pi_\mp q(\xi,\eta)\mu^\pm(\xi,\eta,k), \tag{3}$$

where the linear operator E is defined by

$$[Ef(\cdot)](x-\xi,y) \equiv \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} dm\, \exp[im(x-\xi)J + im(y-\eta)]f(\eta) = f[y+(x-\xi)J],$$

and $f[y+(x-\xi)J]$ denotes the matrix obtained from $f(x)$ by evaluating its lth row at $y+(x-\xi)J_l$. Furthermore from the definition of $\hat{J}$ it follows that $\exp(\hat{J})f = \exp(J)f\exp(-J)$. Equation (3) can be derived from (2) by taking the Fourier transform in the y direction.

With the assumption that the linear integral equations (3) have no homogeneous solutions it follows that μ^+ and μ^- are holomorphic functions of k, for $k_I > 0$ and $k_I < 0$, respectively. Hence the function $\mu(x,y,k)$ is a *sectionally holomorphic function of k having a jump across* $k_I = 0$. Thus $\partial\mu/\partial\bar{k} = 0$ for all k with $k_I \neq 0$ and $\partial\mu/\partial\bar{k} = \mu^+(x,y,k) - \mu^-(x,y,k)$ for $k = k_R$. Manipulating Eqs. (3) we find the following scattering equation:

$$\mu^+(x,y,k) - \mu^-(x,y,k) = \int_{-\infty}^{\infty} dl\, \mu^-(x,y,l)e^{ilJx+ily}f(l,k)e^{-ikJx-iky}, \quad k = k_R. \tag{4}$$

In Eq. (4) the *scattering data* $f(l,k)$ satisfy

$$f(l,k) - \int_{-\infty}^{\infty} dm\, T_+(l,m)f(m,k) = T_+(l,k) - T_-(l,k); \quad l,k \text{ real}, \tag{5}$$

where $T_\pm$ are defined in terms of q via

$$T_\pm(l,k) \equiv (1/2\pi)\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, e^{-ilJ\xi - il\eta}\pi_\pm q(\xi,\eta)\mu^\mp(\xi,\eta,k)e^{ik\xi J + ik\eta}; \quad l,k \text{ real}. \tag{6}$$

I note the remarkable fact that $f(l,k)$ can be solved in *closed form* in terms of T_+, T_-. This is because the kernel of Eq. (5) is strictly upper triangular. For example if $n=2$ then $f_{22}=0$, $f_{21} = -T_+$, $f_{12} = T_-$, and $f_{11}(l,k) = -\int_{-\infty}^{\infty} dm\, T_+(l,m)T_-(m,k)$.

Equation (4) defines a Riemann-Hilbert problem for $\mu(x,y,k)$ in terms of $f(l,k)$. Its solution is given by the following linear integral equation [which can be obtained by taking the "minus" projection of (4)]:

$$\mu^-(x,y,k) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\nu \frac{\mu^-(x,y,l)e^{ilJx}f(l,\nu)e^{-i\nu Jx + i(l-\nu)y}}{\nu - \kappa + i0} = I. \tag{7}$$

By taking the large-k limit of (7) and comparing with that of (2) we find

$$q(x,y) = -(1/2\pi)\hat{J}\int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\nu\, \mu^-(x,y,l)e^{ilJx}f(l,\nu)e^{-i\nu Jx + i(l-\nu)y}. \tag{8}$$

Hence, the solution of the inverse problem of (2) is given by (8), where $\mu^-(x,y,k)$ is obtained from (7) and the scattering data $f(l,k)$ can be found in closed form in terms of $T_\pm(l,k)$ from (5).

(b) I now consider the inverse problem associated with the elliptic system which is obtained from (2) by replacing μ_y by $-i\mu_y$. [J is a constant real diagonal matrix with all its entries different from each other and $q(x,y)$ tends to zero for large x,y.] A matrix eigenfunction $\mu(x,y,k)$ which solves the elliptic system, is bounded for all complex values of k, and tends to I as $k\to\infty$ is defined by the following matrix equation:

$$\{\mu(x,y,k)\}_{ij} = \{I\}_{ij} + (1/2\pi)\left(\int_{-\infty}^{x} d\xi \int_{-\infty}^{C_{ij}k_I} dm \int_{-\infty}^{\infty} d\eta - \int_{x}^{\infty} d\xi \int_{C_{ij}k_I}^{\infty} dm \int_{-\infty}^{\infty} d\eta\right) \times \{e^{(mJ+ik\hat{J})(x-\xi)+im(y-\eta)}q(\xi,\eta)\mu(\xi,\eta,k)\}_{ij} \tag{9}$$

for $J_i > 0$, where $C_{ij} \equiv (J_i - J_j)/J_i$, and for $J_i < 0$ the integrals with respect to dm are replaced by $\int_{C_{ij}k_I}^{\infty} dm$ and $\int_{-\infty}^{C_{ij}k_I} dm$, respectively ($\{f\}_{ij}$ denotes the ijth entry of the matrix f). Comparing (9) to (3) it follows that (a) Eq. (9), in contrast to (3), has no jump across $k_I = 0$; (b) Eq. (9) *depends explicitly on* k_I. Hence the solution $\mu(x,y,k)$, although bounded for all complex values of k, is analytic nowhere with respect to k, since $\partial\mu/\partial\bar{k} \neq 0$ [for simplicity of notation I still write $\mu(x,y,k)$ instead of $\mu(x,y,k_R,k_I)$].

The "departure from holomorphicity" of $\mu(x,y,k)$ is measured by $\partial\mu/\partial\bar{k}$. With the assumption that (9) has no homogeneous solutions (as it was pointed out above this excludes the manifestation of lumps), by differentiation of (9) it follows that $\partial\mu/\partial\bar{k}$ satisfies an equation obtained from (9) by replacing I by $\Omega(x,y,k_R,k_I)$, where $\{\Omega\}_{ii} = 0$,

$$\{\Omega\}_{ij} = T_{ij}(k_R,k_I)\exp[\theta_{ij}(x,y,k_R,k_I)], \quad \theta_{ij}(x,y,k_R,k_I) = iC_{ij}(J_i k_R x + k_I y),$$

$$T_{ij}(k_R,k_I) \equiv (i/4\pi)\,\mathrm{sgn}(J_i)C_{ij}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \{q(\xi,\eta)\mu(\xi,\eta,k)\}_{ij}\exp[-\theta_{ij}(\xi,\eta,k_R,k_I)]. \tag{10}$$

A "$\bar{\partial}$" problem can be defined as follows: Given $\partial\mu/\partial\bar{k}$ find μ. In order to formulate such a problem here, one needs to express $\partial\mu/\partial\bar{k}$ in terms of μ. This relationship is as follows:

$$\frac{\partial\mu(x,y,k)}{\partial\bar{k}} = \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mu\left(x,y,k_R + i\frac{J_j}{J_i}k_I\right)T_{ij}(k_R,k_I)w^{ij}(x,y,k_R,k_I), \tag{11}$$

where w^{ij} is a matrix with zeros everywhere except at its ijth entry which equals $\exp\theta_{ij}(x,y,k_R,k_I)$. Equation (11) can be derived by first introducing an eigenfunction $N^{ij}(x,y,k_R,k_I)$ which satisfies an equation obtained from (9) by replacing I by w^{ij}, and then by using the important "symmetry" condition $N^{ij}(x,y,k_R,k_I) = \mu(x,y,k_R + i(J_j/J_i)k_I)w^{ij}$. Using (11) and the equation

$$f(k) = (2\pi i)^{-1}\int\int_R [\partial f(z)/\partial\bar{z}](z-k)^{-1}dz\wedge d\bar{z} + (2\pi i)^{-1}\int_C f(z)(z-k)^{-1}dz$$

(which is an extension of Cauchy's formula over the region R bounded by the contour C[14]), one obtains the following linear integral equation for $\mu(x,y,k)$:

$$\mu(x,y,k) - \frac{1}{2\pi i}\int\int_{R_\infty}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{z-k}\mu(x,y,z_R + i\frac{J_j}{J_i}z_I)T_{ij}(z_R,z_I)w^{ij}(x,y,z_R,z_I)dz\wedge d\bar{z} = I, \tag{12}$$

where R_∞ is the entire complex z plane and $dz\wedge d\bar{z} = -2i dz_R dz_I$. Once μ is found q can be reconstructed via

$$q(x,y) = \frac{1}{2\pi}\hat{J}\int\int_{R_\infty}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mu\left(x,y,z_R + i\frac{J_j}{J_i}z_I\right)T_{ij}(z_R,z_I)w^{ij}(x,y,z_R,z_I)dz\wedge d\bar{z}. \tag{13}$$

Hence, the solution of the inverse problem of the above elliptic system is given by (13), where $\mu(x,y,k)$ is obtained from (12) and the *scattering data* $T_{ij}(k_R,k_I)$ are defined by (10).

(c) Suppose that an evolution equation for $q(x,y,t)$ possesses a Lax pair[15] (i.e., it can be written as the compatibility condition of two linear equations). If the time-independent part of this Lax pair is contained in (1), then the above results can be directly used to linearize the initial-value problem associated with this equation. To achieve this one only needs to evaluate the evolution of the scattering data (this is straightforward by appropriate use of the time-dependent part of the Lax pair).

The n-wave interaction equations

$$q_{ij_t}=\alpha_{ij}q_{ij_x}+\beta_{ij}q_{ij_y}+\sum_{\substack{k=1\\k\neq j}}^{n}(\alpha_{ik}-\alpha_{kj})q_{ik}q_{kj},\quad i,j,k=1,\ldots,n, \tag{14}$$

are contained in (2), with J defined by $\alpha_{ij}=(C_i-C_j)/(J_i-J_j)$, $\beta_{ij}=C_i-J_i\alpha_{ij}$. Hence the initial-value problem of (14) can be solved through (8), where $f(l,k,0)$ is found from (5), (6), and

$$f(l,k,t)=\exp(iltA_{20})f(l,k,0)\exp(-iktA_{20}),\quad A_{20}=\mathrm{diag}(C_1,\ldots,C_n).$$

We call DS I the set of equations

$$iQ_t+\tfrac{1}{2}(Q_{xx}+Q_{yy})=-\sigma|Q|^2Q+\varphi Q;\quad \varphi_{xx}-\varphi_{yy}=2\sigma(|Q|^2)_{xx},\quad \sigma=\pm 1. \tag{15}$$

These equations are also contained in (2) with $J_1=1$, $J_2=-1$, $q_{12}=Q$, and $q_{21}=\sigma\bar{Q}$. The evolution of $f(l,k,t)$ is given by

$$f(l,k,t)=\exp(-l^2tA_{30})f(l,k,0)\exp(k^2tA_{30}),\quad A_{30}=\mathrm{diag}(i,-i).$$

We call DS II the set of equations obtained from (15) by replacing Q_{xx} by $-Q_{xx}$ and φ_{yy} by $-\varphi_{yy}$. These equations are related to the elliptic case (b) above. Hence the initial-value problem of DS II can be solved through (13) where the scattering data $\Omega^{ij}\equiv T_{ij}w^{ij}$ is computed via (10) and

$$\Omega^{ij}(x,y,k_R,k_I,t)=\exp(\hat{k}^2A_{30}t)\Omega^{ij}(x,y,k_R,k_I,0)\exp(-k^2A_{30}t),$$

$\hat{k}\equiv k_R+i(J_j/J_i)k_I$.

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