Factorized S-Matrix Method for the Thermodynamics of the Massive Thirring Model

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A new method based on the S-matrix factorization is developed for the thermodynamics of the massive Thirring model. The method can be applied to any charge sector and coupling constant. In the zero-charge sector, the present theory reproduces all the Bethe-*Ansatz*-theory results in the attractive-coupling regime, and agrees with a perturbation calculation in the repulsive-coupling regime.

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The Bethe *Ansatz* (BA) method developed earlier by Yang and Yang,[1] Gaudin,[2] and Takahashi and Suzuki[3] has long been the only method for studying the thermodynamics of the completely integrable models in one dimension. Recent successful applications of this method include those to the Kondo problem[4] and the massive Thirring model (MTM).[5,6] All of these applications[3-6] require the introduction of certain exotic excitations, i.e., the strings in the Heisenberg-Ising ring[3] and Kondo problem,[4] and the Korepin excitations in the MTM.[5] Although these exotic excitations play important roles in the thermodynamics, their physical meanings have not been understood. Recently, Chung and Chang[6] pointed out the unobservability of the Korepin excitations in the MTM. In this Letter, we present a new approach to the MTM thermodynamics based on the S-matrix factorization theory.[7,8] Because of the generality of the MTM, the present formulation is readily applicable to any completely integrable system for which factorized S matrices are known. Unlike the BA theory, the present theory does not invoke the Korepin excitations. Instead, the soliton-antisoliton (s-$\bar{s}$) backscattering effect is included explicitly in the construction of the eigenstate from the beginning. Our theory is applicable to arbitrary charge sector and arbitrary coupling constant.[9] For special cases such as the zero-charge sector with coupling constants $P_0=\nu_1$ and ν_1+1/ν_2, where ν_1 and ν_2 are integers greater than 2, the present theory reproduces all the BA-theory results.[5,6] The theory is also shown to agree with a perturbational calculation in the repulsive-coupling regime.

Our main task here is to quantize multi-(s-$\bar{s}$) excitations. Let us consider a MTM system with N_s solitons and $N_{\bar{s}}$ antisolitons. According to the S-matrix factorization theory, the interaction of these particles accompanies no particle production or annihilation and can be described by the product of successive two-body scatterings. Therefore, the energy eigenstate of the N ($=N_s+N_{\bar{s}}$)-particle system can be expressed as[10]

$$|\psi\rangle=\int\cdots\int\prod_{i=1}^{N}dX_i\sum_{Q\in S_N}\theta(X_Q)\exp(i\sum_j P_jX_j)\Phi_Q\prod_{i=1}^{N}\varphi^{\dagger}(X_i)|0\rangle, \quad (1)$$

where $\varphi^{\dagger}(X)$ creates a particle (either soliton or antisoliton) at X, and Q is an element of the permutation group S_N for a set of integers $\{1,2,\ldots,N\}$. The function $\theta(X_Q)$ equals 1 when $X_{Q1}<X_{Q2}<\ldots<X_{QN}$ and 0 otherwise. The Φ_Q represents a linear combination of ${}_NC_{N_s}$ basis charge states. Following the procedures of Yang,[11] we can describe the quantization of the set of momenta $\{P_j\}$ as the eigenvalue problem

$$Z_j\Phi_I=\exp(iP_jL)\Phi_I \quad \text{for } j=1,2,\ldots,N, \quad (2)$$

where $Z_j\equiv S_{j+1,j}S_{j+2,j}\cdots S_{N,j}S_{1,j}\cdots S_{j-1,j}$, L is the length of the system, and the subscript I denotes the identity permutation. The two-body S matrix is defined by[8] $S_{ij}|ij\rangle=u_{ij}|ij\rangle$ and $S_{ij}|\bar{i}\bar{j}\rangle=t_{ij}|\bar{i}\bar{j}\rangle+r_{ij}|i\bar{j}\rangle$, where i ($\bar{i}$) in the ket vector denotes the soliton (antisoliton) with rapidity α_i. $X_{ij}=X(\alpha_i-\alpha_j)$ for $X=u$, t, and r representing the scattering amplitudes for s-$s(\bar{s}$-$\bar{s})$, s-$\bar{s}$ forward, and s-$\bar{s}$ backward scatterings, respectively. The soliton (antisoliton) momentum is related to its rapidity by $P=M_s\sinh\gamma\alpha$, where $M_s=\mu/\omega$ is the zero-temperature soliton mass in units of the zero-temperature free ($P_0=2$) soliton mass, and $\gamma=\pi/2\mu$ with $\omega=\pi/P_0$ and $\mu=\pi-\omega$.[6] With the factorization conditions satisfied by the two-body S matrices,[8] we can show that the Z_j's commute, and therefore they can be simultaneously diagonalized. Yang's method was recently applied to the Kondo problem,[4] where the eigenstate of the bare Hamiltonian was consid-

ered. The eigenvalue problem similar to Eq. (2) for the spin state was derived and eventually reduced to the BA formalism. In contrast to this, our argument is on the renormalized eigenstate (1) with knowledge of the factorized S matrix. The quantization of soliton and antisoliton momenta can be derived directly from the eigenvalue problem (2) to a thermodynamically desirable accuracy without resorting to the BA formalism.

The eigenvalue problem (2) allows ${}_NC_{N_s}$ charge eigenstates, with each of which is associated a unique set $\{\exp(iP_jL);\ j=1,2,\ldots,N\}$. Let us consider the thermodynamic limit $N\to\infty$ and $L\to\infty$, keeping N/L finite. Then the relevant microscopic energy eigenstates have the property that $\alpha_i-\alpha_{i-1}\sim O(1/L)$ for the overwhelming majority of i, and the corresponding macroscopic state is described by the soliton and antisoliton densities, $\rho_s(\alpha)$ and $\rho_{\bar{s}}(\alpha)$, as well as the associated hole densities, $\tilde{\rho}_s(\alpha)$ and $\tilde{\rho}_{\bar{s}}(\alpha)$. As in the BA formalism, we now assume that the correlation length in the rapidity space is much shorter than the length scale $\Delta\alpha$, satisfying $1/L\ll\Delta\alpha\ll 1$ [e.g., $\Delta\alpha\sim O(1/\sqrt{L})$]. Under this assumption, the total entropy of the system can be expressed as a sum of local entropies in the rapidity space. The same assumption simplifies the eigenvalue problem (2) and makes it solvable for $\{P_j\}$ with an accuracy of $O(1)$. To show this, we divide the rapidity space into patches of length $\Delta\alpha$, and make the corresponding grouping of the two-body S matrices, namely $Z_j=\prod_{k=1}^{n}Z_{kj}$, where $n\sim N/L\Delta\alpha$ and Z_{kj} is the product of $\sim L\Delta\alpha$ two-body S matrices. Under the above assumption the eigenstate Φ_I of Eq. (2) becomes the simultaneous eigenstate of the Z_{kj}'s. Since the degree of accuracy for this procedure is the same as for calculating the total entropy, this procedure is expected to be correct up to $O(1)$ for $\{P_j\}$, which is sufficient for the thermodynamics. The eigenvalue problem (2) is then reduced to

$$Z_{kj}\Phi_I=\lambda_{kj}\Phi_I, \tag{3}$$

with $\exp(iP_jL)=\prod_{k=1}^{n}\lambda_{kj}$. We can further simplify the reduced eigenvalue problem (3) by replacing all the rapidities in the relevant patch with a representative one, say α', thereby neglecting $O(1/L)$ quantities. With this simplification all λ_{kj}'s in Eq. (3) reduce to a function of macroscopic quantities $\rho_s(\alpha')$ and $\rho_{\bar{s}}(\alpha')$ of the kth patch, and the resulting charge eigenstates transform into each other under the exchange of particles in the kth patch. This means that any charge eigenstates so obtained are also the eigenstates of any operators which are produced from Z_{kj} by permutations of the two-body S matrices in Z_{kj}. Thus, any two S matrices in Z_{kj} commute and Φ_I becomes the simultaneous eigenstate of all S matrices in Z_{kj}. The eigenvalues of the two-body S matrix are u and $S_\pm=t\pm r$ for the scatterings between s-$s(\bar{s}$-$\bar{s})$ and s-$\bar{s}$ with even (+) and odd (-) parities.[8] How frequently u or $S_\pm$ appears in the eigenvalue is determined by the probabilities for s-s, $\bar{s}$-$\bar{s}$, and s-$\bar{s}$ scatterings. Consider the jth patch denoted by a representative rapidity α, to which the jth particle belongs; then the eigenvalue for the scattering between the kth and jth patches is given by

$$\exp(i\phi_{kj})=u(\alpha'-\alpha)^{y(L\Delta\alpha)^2}[S_+(\alpha'-\alpha)S_-(\alpha'-\alpha)]^{z(L\Delta\alpha)^2/2}, \tag{4}$$

with

$$y=\rho_s(\alpha)\rho_s(\alpha')+\rho_{\bar{s}}(\alpha)\rho_{\bar{s}}(\alpha'),$$

$$z=\rho_s(\alpha)\rho_{\bar{s}}(\alpha')+\rho_{\bar{s}}(\alpha)\rho_s(\alpha'),$$

where we have assumed that the parity-(±) scatterings occur with an equal probability. From Eq. (4) we finally find that the quantization of soliton and antisoliton momenta associated with the jth patch is given, up to $O(1)$ accuracy, by

$$P_j(\alpha)=(2\pi/L)\times\text{integer}+\sum_{i=s,\bar{s}}\Delta_{ji}*\rho_i \tag{5}$$

($j=s$ or $\bar{s}$), where $\Delta_{ss}=\Delta_{\bar{s}\bar{s}}\equiv -i\ln u$, $\Delta_{s\bar{s}}=\Delta_{\bar{s}s}\equiv -i\ln(S_+S_-)^{1/2}$. Here we have introduced a convenient notation $a*b\equiv\int_{-\infty}^{\infty}d\alpha'\,a(\alpha'-\alpha)b(\alpha')$.

The remaining procedures in developing the MTM thermodynamics are to include the contributions of breathers, to express the free energy as a functional of ρ_j and $\tilde{\rho}_j$, and to minimize it with respect to variations of ρ_j, where j runs over soliton, antisoliton, and breathers. These procedures have been well established in the BA formalism. One of the novel features of the present method is that it can be applied to the charged sector as well as the zero-charge sector. Here we minimize the free energy under the constraint of constant net charge, i.e.,

$$L\int_{-\infty}^{\infty}[\rho_s(\alpha)-\rho_{\bar{s}}(\alpha)]\,d\alpha=N_s-N_{\bar{s}}.$$

Consider the case $P_0=\nu_1+q$ with ν_1 an integer ≥ 1 and $0<q\leq 1$.[9] Then there are ν_1-1 kinds of breathers with the physical momenta

$$P_j=2M_s\sin[(j\pi/2)(2\gamma-1)]\sinh(\gamma\alpha),$$

$j=1,2,\ldots,\nu_1-1$.[12] Letting i and j run over soli-

ton, antisoliton, and breathers, we have obtained the following basic equations for ρ_j's and $\tilde{\rho}_j$'s:

$$\frac{dP_j}{d\alpha} = 2\pi(\rho_j + \tilde{\rho}_j) + \sum_i \frac{\partial}{\partial\alpha} \Delta_{ji} * \rho_i , \tag{6}$$

$$\epsilon_j = \tilde{E}_j + \frac{T}{2\pi} \sum_i \frac{\partial}{\partial\alpha} \Delta_{ij} * \ln\left[1 + \exp\left(-\frac{\epsilon_i}{T}\right)\right], \tag{7}$$

where we have defined $\tilde{\rho}_j/\rho_j = \exp(\epsilon_j/T)$ and $\tilde{E}_j = E_j - A\delta_{js} + A\delta_{j\bar{s}}$. $E_j = (1/\gamma)dP_j/d\alpha$ represents the particle energy, A is a Lagrange multiplier, and δ is the Kronecker delta symbol. A compact expression for the free energy density can be derived from Eqs. (6) and (7) as

$$F = -\frac{\gamma T}{2\pi} \sum_i \int_{-\infty}^{\infty} d\alpha\, E_i \ln\left[1 + \exp\left(-\frac{\epsilon_i}{T}\right)\right] + A\int_{-\infty}^{\infty} d\alpha\,(\rho_s - \rho_{\bar{s}}) . \tag{8}$$

Solving the coupled integral equations (6) and (7) for a given A and T, we can find the soliton, antisoliton, and breather densities. Thus Eq. (8) gives us the free energy and hence all the thermodynamic quantities.

Two remarks are in order. First, the chemical potentials of particles are calculated from the formula $\mu_j = (\partial F/\partial N_j)_{L,T,N_{i\neq j}}$, where N_j represents the number of the jth particle. We obtain $\mu_j = A$, $-A$, and 0, for j denoting soliton, antisoliton, and breathers, respectively. In particular, in the zero-charge sector $\epsilon_s = \epsilon_{\bar{s}}$ by symmetry, and hence $A = 0$ from Eq. (7). In the previous BA formalism,[5,6] it was implicitly assumed that the chemical potentials of particles are all zero in the zero-charge sector. Here this assumption has been given a firm basis. Second, in the BA formalism the $\epsilon_j(\alpha)$ represent the excitation spectrum.[1,6] This argument remains valid in the present formalism. Then from Eq. (7) it is seen that when A is positive (negative) the soliton mass is lighter (heavier) than the antisoliton mass. Detailed analysis of the charged-sector thermodynamics will be presented elsewhere.

Finally, we show that in the zero-charge sector the present theory reproduces all the BA-theory results in the attractive-coupling regime, and gives results in agreement with a perturbational calculation in the repulsive-coupling regime. We have numerically solved Eq. (7) for the $A = 0$ case and calculated the coupling-constant dependences of the free energy F, lowest-breather mass M_{1b}, and soliton mass M_s. In Fig. 1, the solid lines represent results from the present theory at $T = 2$, whereas the crosses are from the BA theory.[6] We find that the agreement is exact within the numerical precision. In fact, at the points $P_0 = \nu_1 \pm 0$ ($\nu_1 \geq 2$) the equivalence of both theories can be proved analytically. On the other hand, we have performed a first-order perturbation calculation for the free energy in the repulsive-coupling regime $g = 2\mu - \pi < 0$. Note that the soliton mass M_s is coupling-constant dependent; so is the free energy of the free soliton and antisoliton system with mass renormalization. Let us write the other finite-coupling correction to the free energy as ΔF. Then to first order in g, ΔF is given by the Matsubara formula as $\Delta F = \langle H^{\rm int}\rangle_0$, where $\langle\cdots\rangle_0$ denotes the thermal average with respect to the unperturbed system. Writing $\Delta F = F_+ + F_-$, where F_+ and F_- represent the s-$s(\bar{s}$-$\bar{s})$ and s-$\bar{s}$ scattering contributions, respectively, we have

$$F_\pm = -\frac{g}{4\pi^2} \iint d\alpha\, d\alpha'\, f(\alpha) f(\alpha')[1 \mp \cosh(\alpha - \alpha')], \tag{9}$$

where $f(\alpha) = 1/[1 + \exp(\cosh\alpha/T)]$. On the other hand, we have solved Eq. (7) for ϵ_s with $A = 0$ to first order in g and evaluated the finite-coupling

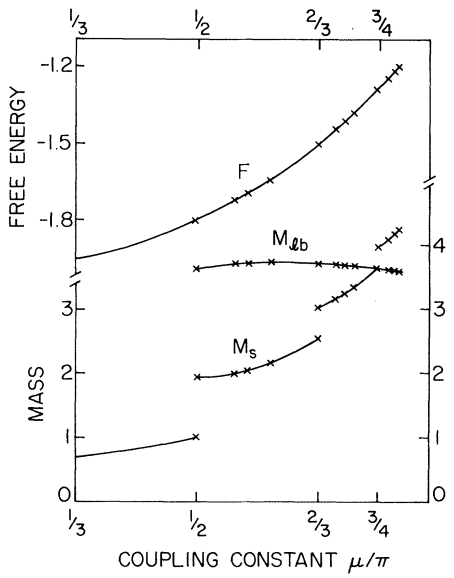

FIG. 1. Free energy F, soliton mass M_s, and lowest-breather mass M_{1b} as functions of the coupling constant μ/π at $T = 2$ (solid curves). Also included are results from the BA theory (crosses) (Ref. 6). The mass and temperature are measured in units of the zero-temperature free-soliton mass.

correction to the free energy by Eq. (8). We find that, apart from a term due to the coupling-constant dependence of the soliton mass, this correction comprises contributions from the s-$s(\bar{s}$-$\bar{s})$ and s-$\bar{s}$ scatterings, which are identical to F_+ and F_- given by Eq. (9), respectively.

In summary, we have developed a new method for the MTM thermodynamics. This method is quite general in that it applies to arbitrary charge sector and arbitrary coupling constant. In the zero-charge sector, the present theory reproduces all the BA theory results in the attractive-coupling regime, and yields results in agreement with first-order perturbation theory in the repulsive regime. Moreover, because of the generality of the MTM and the simplicity of this method, it can be readily applied to any completely integrable system for which the factorized S matrices are known.

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