Volkov Solutions, Gauge-Poincaré Transformations, and Plane-Wave Decoupling

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(Received 11 July 1983)

A solution is found for the vector-particle equation with Yang-Mills coupling to an electromagnetic plane-wave potential. The local gauge and Poincaré transformation structure of such generalized Volkov solutions of single-particle equations (spin ≤ 1) leads to a decoupling theorem for the scattering of a system of particles immersed in an external plane wave.

PACS numbers: 11.15.Kc, 11.10.Qr, 41.70.+t

The Volkov solutions of the Dirac and Klein-Gordon equations in the presence of an external electromagnetic plane wave are well known and widely used.[1] Nevertheless, their underlying symmetry[2] has not been fully exploited nor has a spin-1 version been given.

In this Letter we find a counterpart to the Volkov solution in the vector-particle case for an external Yang-Mills coupling. All three Volkov wave functions (spin $0,\frac{1}{2},1$) are shown to be generated by a field-dependent, local gauge-Poincaré transformation of the free-field solutions. The transformation structure is intimately tied to the local gauge invariance of renormalizable field theories for spin; a new characteristic of local gauge symmetry thus emerges.

With the form of such solutions we derive formulas for the collective modifications of charged-particle lines due to background plane waves. We prove a decoupling theorem for tree-graph amplitudes for particle interactions that take place in this background, namely that the collective effects vanish in certain kinematical zones, as long as any derivative couplings are "minimal."[3,4] In lowest order this reduces to the radiation zeros found in single-photon tree amplitudes.[4,5] The decoupling effect can be included among interesting possibilities for laser/plasma experiments such as the intensity-dependent frequency shift.[6]

Consider a particle with charge Q and mass m, gauge-covariantly coupled to an external plane wave $A_\mu = A_\mu(n\cdot x)$, $n^2=0$, in the Lorentz gauge $n\cdot A=0$. The corresponding wave equations for the three spins are

$$(D^2+m^2)\Psi=0 \text{ (scalar)}; \tag{1a}$$

$$(i\not{D}-m)\Psi=0 \text{ (Dirac)}; \tag{1b}$$

$$\left.\begin{aligned}&(D^2+m^2)\Psi_\mu+2iQ\,F_{\mu\nu}\Psi^\nu=0,\\ &D\cdot\Psi=0 \text{ (vector)}.\end{aligned}\right. \tag{1c}$$

The covariant derivative is $D\equiv\partial+iQA$ and $F_{\mu\nu}$ is the field-strength tensor.

We find that the solutions to (1) can all be written in the form

$$\Psi(x)=ULT\chi(x), \tag{2}$$

where χ is the free solution ($Q=0$) and ULT is a product of local gauge (U), Lorentz (L), and displacement (T) transformations. For the respective {scalar; Dirac; vector} plane-wave solutions

$$\chi_p(x)=e^{-ip\cdot x}\{1; w(p); \eta(p)\}, \tag{3a}$$

with

$$p^2=m^2,\quad \not{p}w=mw,\quad \eta\cdot p=0, \tag{3b}$$

we have

$$L=e^S=\{1;\,1+\mathfrak{F};\,\Lambda\}, \tag{4a}$$

$$S=\{0;(Q/2n\cdot p)\not{n}\not{A}\equiv\mathfrak{F};\ (Q/n\cdot p)(n_\mu A_\nu-A_\mu n_\nu)\equiv\Omega_{\mu\nu}\}, \tag{4b}$$

representing an element of the local little group $E_2(n)$:

$$\Lambda_{\mu\nu}=(e^\Omega)_{\mu\nu} = g_{\mu\nu}+\Omega_{\mu\nu}-[Q^2/2(n\cdot p)^2]A^2n_\mu n_\nu. \tag{5}$$

Also

$$U(\theta)=e^{iQ\theta},\quad \theta=(Q/2n\cdot p)\int^{n\cdot x}dz\,A^2(z), \tag{6}$$

and

$$T(d)=e^{-ip\cdot d},\quad d^\mu=(Q/n\cdot p)\int^{n\cdot x}dz\,A^\mu(z). \tag{7}$$

The demonstration of (2) for the plane waves (3) follows from Lorentz covariance and the important operator identity

$$(UT)^{-1}D_\mu(UT)=\Lambda_\mu{}^\nu\partial_\nu. \tag{8}$$

This identity, which holds when we can freely interchange $p\leftrightarrow i\partial$, shows a fundamental conspiracy among internal and space-time symmetries where the phase transformations change the covariant derivative into the Lorentz transformation of the free derivative.

The preceding results generalize the form of the Volkov solution for a Dirac particle given long ago by Taub[2] in terms of "variable Lorentz matrices." Our extension to the vector case as well as our identification of the universal nature of the gauge-Poincaré parameters (θ, d, Λ) obtain only for minimal derivative couplings.[3] The universality of (2) will be lost, for example, for solutions with anomalous magnetic moments.

In the absence of an external field, the tree amplitude for the scattering of a system of particles (spins ≤ 1) can be written as

$$\mathcal{T} = \Pi_V \Pi_I \int dp_I \, D(p_I) V(k), \tag{9}$$

where the k-legged vertices $V(k)$ include momentum-conservation delta functions $\delta(\sum^k p_i)$ and the $D(p_I)$ are internal line propagators. (For a given vertex all particles can be defined as incoming.) In the presence of a minimally coupled external electromagnetic field $A(n \cdot x)$, the corresponding tree amplitude $\mathcal{T}_A$ requires modifications of the external and internal legs at each vertex. For constant particle couplings, these modifications of (9) follow from the Fourier transform of (2) and amount to making the δ-function replacements $\delta \to \delta_{\text{ext}}$, where

$$\delta_{\text{ext}} = \prod_{j=1}^{k} (ULT)_j \delta(\sum^k p_i), \tag{10}$$

along with the substitution $n \cdot x \to in \cdot \partial/\partial p_j$ in each $(ULT)_j$. The additional changes for vertices with derivative couplings are discussed below.

The particle scattering amplitude in the presence of the external field is thereby expressed, to all orders, as a nonlocal momentum-space transformation (derived from a local configuration-space transformation) on the field-free amplitude. For example, the nonlocal operator sequence

$$(e^{\pm q \cdot \partial/\partial p})^l \delta(p) = \delta(p \pm lq) \tag{11}$$

produces the harmonics that are expected for a monochromatic wave, $A_\mu = 2\,\mathrm{Re}\,(N\epsilon_\mu e^{-iq\cdot x})$, with frequency ω and momentum $q = \omega n$. Strictly, a spatial cutoff on A is needed, a point to which we will return.

We now explore the condition under which all $(ULT)_j$ refer to the same group parameters such that their product in (10) collapses to unity by invariance. Suppose that

$$Q_i / n \cdot p_i = \text{same} \tag{12}$$

for all external particles i (charge Q_i, momentum p_i), a constraint which specializes to the so-called null-zone condition[7,8] for radiation zeros. Then θ_j, d_j, and Λ_j are all independent of j since the action of $\partial/\partial p_j$ on the delta function in (10) is the same for all j. For each vertex, therefore, charge conservation, momentum conservation, and Lorentz invariance yield

$$\prod_{j=1}^{k} (ULT)_j = 1, \quad \text{null zone}, \tag{13}$$

which is the statement of the decoupling.

According to (13) the null zone is forbidden to the particles, unless they can get there by collisions among themselves (an "elastic" limit). Because of the gauge nature of its interactions, an external plane wave can transform itself away from a system of particles. The system, as with Perseus in his Helmet of Hades, becomes invisible in the "elastic" null zone.

For example, consider electron scattering, $e^- + e^- \to e^- + e^-$, in the presence of an electromagnetic wave train perpendicular to the incident (c.m.) electron beams. In first-order photon exchange, but to all orders in the external field, the only events where the electrons have equal energies and equal angles relative to the plane-wave axis [satisfying (12)] are the elastic forward-scattering collisions and these are without external-field modulation. The inelastic events require interaction with the plane wave which, for a monochromatic wave, corresponds to the emission/absorption of a "photon" with momentum lq, $l = 1, 2, \ldots$.

We can extend the decoupling theorem to particle interactions that include minimal derivatives.[3] The fundamental identity (8) indicates that single derivatives of scalar fields are Lorentz transformed as a result of the plane-wave interaction so that Lorentz invariance again leads to (13). Derivatives of Dirac and vector fields produce extra terms in (10), involving derivatives of L, that have no general mechanism for their cancellation in the null zone. However, the trilinear Yang-Mills interaction for vector fields X, Y, Z,

$$\mathcal{L}_{\text{YM}} = X^\mu (Y^\nu \overleftrightarrow{D}_\mu Z_\nu) + Y^\mu (Z^\nu \overleftrightarrow{D}_\mu X_\nu) + Z^\mu (X^\nu \overleftrightarrow{D}_\mu Y_\nu) + \text{H.c.}, \tag{14}$$

does yield a cancellation of the L derivatives, following from the Bianchi identity for $F_{\mu\nu}$, so that the decoupling remains true in this case. This also implies that Higgs-like second derivatives of scalar fields are allowed corresponding to the replacement of any of the vector fields in (14) by a gradient of a scalar field.

Summary and remarks.—(1) Our principal results—the form and spin-1 extension shown in (2), the identity (8), the external-field modification rule (10), and the decoupling theorem (13)—are distinctively related to renormalizable gauge theories in view of the minimal-derivative requirement.

(2) Our results provide a basis[9] for the developments in Refs. 4 and 5. The internal-line decomposition identity, vertex expansion, and radiation representation are generalized by the exponential forms in (10), while Eq. (8) explains the relationship seen between first-order contact currents and the first-order Lorentz transformation. Radiation symmetry now reads $Q_i/n\cdot p_i \to Q_i/n\cdot p_i + C$, which follows from (10) and (13), and there is also a generalization available for a non-Abelian background wave. The amplitudes for N collinear photons can be shown to be proportional to the Q^N terms derived from (10); thus multiphoton zeros also follow from the decoupling theorem.

(3) To be precise, the S-matrix description requires that the wave train $A(n\cdot x)$ is finite, so that the monochromatic external field must be taken as a limiting case.[10] Although, for example, the intensity-dependent frequency shift is found only when this limit is very carefully considered, the decoupling result is independent of the limiting procedure, surviving careful analysis such as that given in the papers of Brown and Kibble.[6] Indeed, we find that *the frequency shift vanishes in the null zone* as it must for decoupling.[11]

(4) Decoupling does not hold in general for nonminimal interactions, which spoil universality; for closed loops, where (12) cannot be maintained; and for higher spins [higher powers in (4)] with no mechanism for the cancellation of the extra terms that arise in (10). With additional or different symmetries, higher spins and/or other potentials may have an analogous duality (and ULT transformations) where the interactions are written in terms of (perhaps ultimately all of) the associated symmetries. We might expect that plane waves are necessary, however, for the perfect interference shown in the decoupling.

We are grateful to Stanley Brodsky for discussions and encouragement. This work was supported by the National Science Foundation.

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[1]D. M. Volkov, Z. Phys. 94, 250 (1935). Recent references and applications can be traced from, for example, J. Bergou and F. Ehlotzky, Phys. Rev. A 27, 2291 (1983).

[2]The form of the Volkov solution most relevant to our work is given by A. H. Taub, Rev. Mod. Phys. 21, 388 (1949), and Phys. Rev. 73, 786 (1948). References and a useful study based on the symmetry of the field-strength tensor are found in J. Kupersztych, Phys. Rev. D 17, 629 (1978), and 24, 873 (1981).

[3]The following kinds of (covariant) derivatives are allowed: None for Dirac, single for scalar, single for vector and double (Higgs-like) for scalar as arise in Yang-Mills trilinear forms, and products of these forms. This set is termed "gauge theoretic" in Ref. 4.

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[5]The first example of such zeros was found in weak-boson physics by K. O. Mikaelian, M. A. Samuel, and D. Sahdev, Phys. Rev. Lett. 43, 746 (1979) [see also R. W. Brown, D. Sahdev, and K. O. Mikaelian, Phys. Rev. D 20, 1164 (1979)], the factorization algebra for which is given by C. J. Goebel, F. Halzen, and J. P. Leveille, Phys. Rev. D 23, 2682 (1981) [see also Zhu Dongpei, Phys. Rev. D 22, 2266 (1980)]. A review and a discussion of "radiation symmetry" is contained in R. W. Brown, in Proceedings of the Europhysics Study of Electroweak Effects at High Energies, Erice, Italy, February 1983 (to be published).

[6]N. D. Sengupta, Bull. Calcutta Math. Sco. 39, 147 (1947); L. S. Brown and T. W. B. Kibble, Phys. Rev. 133, A705 (1964); T. W. B. Kibble, Phys. Rev. 138, B740 (1965). For a summary, see J. H. Eberly, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1968), Vol. 7, p. 359.

[7]We shall henceforth refer to the region of momentum space over which (12) is satisfied as the (generalized) null zone. The studies in Ref. 4 and in M. A. Samuel, Phys. Rev. D 27, 2724 (1983); G. Passarino, Nucl. Phys. B224, 265 (1983); M. A. Samuel, A. Sen, G. S. Sylvester, and M. L. Laursen, Phys. Rev. D (to be published); S. G. Naculich, Phys. Rev. D 28, 2297 (1983), thus furnish null-zone solutions for a given Fourier component.

[8]Of course, this region may be unphysical (e.g., the charges may not be of the same sign). Since $Q/n\cdot p \neq 0$ requires a neutral particle to be massless and to travel in the wave direction ($p\propto n$), the associated corrections (4)–(7) vanish in the null zone. An exception is a forward vector transition such as Compton scattering, $\gamma e\to\gamma e$, which would not decouple from a background plane wave (cf. Ref. 4).

[9]R. W. Brown and K. L. Kowalski, to be published.

[10]It is shown by Kibble, Ref. 6, that there is a change

in the in-state and out-state phases unless the external field-strength tensor has no zero-frequency component [i.e., unless $A(+\infty) - A(-\infty) = 0$]. The framework for a pure monochromatic wave chosen at the outset is given by Z. Fried and J. H. Eberly, Phys. Rev. 136, B871 (1964); J. H. Eberly and H. R. Reiss, Phys. Rev. 145, 1035 (1966).

[11]The frequency is proportional to $e/n \cdot p' - e/n \cdot p$ for $e^-(p) \rightarrow e^-(p') + \gamma$ in the presence of the external wave. More generally all changes in the particle kinematics due to a finite wave train vanish in the null zone (Ref. 9).