# Some Inequalities among Hadron Masses

Edward Witten
*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544*
(Received 19 October 1983)

Some rigorous inequalities governing hadron masses in QCD are proved. One states that the electromagnetic mass shift of the pion is positive. The other states that if $m_{A\overline{B}}$ is the mass of the lightest meson made from a quark of type $A$ and an antiquark of type $B$, then—under conditions such that annihilation into gluons can be ignored—$2m_{A\overline{B}} \geq m_{A\overline{A}} + m_{B\overline{B}}$. These inequalities agree with experimental data and have analogs for arbitrary vectorlike gauge theories. The first inequality has applications to the vacuum alignment problem in vectorlike technicolor theories.

PACS numbers: 12.35.Eq, 12.70.+q, 13.40.Dk

Recently, there has been a surprising amount of progress in proving rigorous results about quantum chromodynamics. No doubt the most startling property of QCD is color confinement. Tomboulis[1] has proved by a rather intricate although conceptually simple argument that lattice SU(2) gauge theory of arbitrarily weak coupling is confining.

It has been found more recently that surprisingly simple arguments yield rigorous information about the pattern of breaking of global symmetries. Weingarten proved[2] that (at $\theta=0$) of all color-singlet channels with nonzero isospin, the lowest threshold is in the pseudoscalar channel. If one assumes confinement, then this together with 't Hooft's old argument[3] shows that QCD with zero-bare-mass quarks must have a massless pseudoscalar of isospin 1—presumably a Goldstone boson. Vafa and Witten[4] showed that in vectorlike gauge theories (like QCD) vectorlike symmetries (like isospin or baryon number) cannot be spontaneously broken (at $\theta=0$). The combined results of Refs. 2 and 4 strongly indicate that in arbitrary vectorlike theories the conventional wisdom is valid: The axial symmetries are spontaneously broken, and the vector symmetries are unbroken. This proposal was apparently first made explicitly by Peskin.[5]

Nussinov[6] has also argued for some highly plausible but not fully rigorous inequalities among meson and baryon masses. In fact, the second inequality in this paper is a special case of one of Nussinov's inequalities.

In this paper some new inequalities will be proved. The approach will follow the strategy introduced independently in Refs. 2 and 4. The Euclidean fermion determinant in vectorlike gauge theories is positive definite, and so the effective measure

$$d\mu = Z^{-1} \prod_{x,\mu,a} dA_\mu{}^a(x) \det(\not{D}+M) \exp[-(1/4g^2)\int d^4x \,\mathrm{Tr}F_{\mu\nu}{}^2] \tag{1}$$

for the $A_\mu{}^a$ integration obtained after integrating out the fermions is positive definite (at $\theta=0$). Inequalities that hold pointwise continue to hold after integrating with respect to a positive measure, so that any inequality among matrix elements that holds after performing the Fermi integral in a fixed background gauge field holds in the exact theory.

Now, let us consider the electromagnetic mass shift of the charged pions. According to the classical current algebra formula,[7] this is

$$m_{\pi^+}{}^2 - m_{\pi^0}{}^2 = (e^2/F_\pi{}^2)\int (d^4k/k^2)[\langle V_\mu{}^3(k)V_\mu{}^3(-k)\rangle - \langle A_\mu{}^3(k)A_\mu{}^3(-k)\rangle]. \tag{2}$$

Here $V_\mu{}^3$ and $A_\mu{}^3$ are $\bar{q}i\gamma_\mu T^3 q$ and $\bar{q}i\gamma_\mu\gamma_5 T^3 q$, respectively. [The factor of $i$ in the definition of $V_\mu{}^a$ and $A_\mu{}^a$ may seem unfamiliar. It arises as follows. The Hermitian vector current in Minkowski space is $V_\mu{}^a = \bar{\psi}\gamma_\mu\lambda^a\psi$, where $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$ and $\eta_{\mu\nu}$ has signature $(+---)$. If our Euclidean gamma matrices obey $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$, then we must take $\gamma_\mu \to i\gamma_\mu$ in rotating from Minkowski space to Euclidean space.] The two-point functions $\langle V_\mu{}^3(k)V_\mu{}^3(-k)\rangle$ and $\langle A_\mu{}^3(k)A_\mu{}^3(-k)\rangle$ are both positive because $V_\mu{}^3$ and $A_\mu{}^3$ are Hermitian operators. The sign difference $\langle V_\mu{}^3(k)V_\mu{}^3(-k)\rangle - \langle A_\mu{}^3(k)A_\mu{}^3(-k)\rangle$ is not obvious, but on various phenomenological grounds[7,5,8] it has been claimed that (2) is positive in nature and in arbitrary vectorlike theories. In fact, we will prove that for any (Euclidean) $k_\mu$,

$$\langle V_\mu{}^3(k)V_\mu{}^3(-k)\rangle - \langle A_\mu{}^3(k)A_\mu{}^3(-k)\rangle \geq 0, \tag{3}$$

showing that the electromagnetic mass shift of the pion is positive, in agreement with observation.

Specifically, we introduce an arbitrary quark bare mass $m$ and an arbitrary suitable ultraviolet cutoff $\Lambda$, such as the cutoff of Asorey and Mitter,[9] and we work in an arbitrary finite volume $V$. We will prove that (3) holds for any values of $m$, $V$, and $\Lambda$ and therefore in any limit such as $m \to 0$, $V \to \infty$, $\Lambda \to \infty$.

We have

$$\langle V_\mu^3(k)V_\mu^3(-k)\rangle - \langle A_\mu^3(k)A_\mu^3(-k)\rangle$$
$$= (1/V)\int d^4x \int d^4y\, e^{ik\cdot(x-y)}[-\langle \bar{q}\gamma_\mu T^3 q(x)\bar{q}\gamma_\mu T^3 q(y)\rangle + \langle \bar{q}\gamma_\mu\gamma_5 T^3 q(x)\bar{q}\gamma_\mu\gamma_5 T^3 q(y)\rangle]$$
$$= (2/V)\int d\mu \int d^4x\, d^4y\, e^{ik\cdot(x-y)}[+\mathrm{Tr}\,\gamma_\mu S(x,y;m)^A\gamma_\mu S(y,x;m)^A - \mathrm{Tr}\,\gamma_\mu\gamma_5 S(x,y;m)^A\gamma_\mu\gamma_5 S(y,x;m)^A]. \quad (4)$$

[A factor of $(-1)$ reflects Fermi statistics, and a factor of 2 reflects the fact that $\mathrm{Tr}T_3^2 = 2$.] Here $S(x,y;m)^A$ is the propagator from $y$ to $x$ of a fermion of bare mass $m$ in a background gauge field $A$. It may be written

$$S(x,y;m) = \langle x|1/(\not{D}+m)|y\rangle = \langle x|(-\not{D}+m)/[-(\not{D})^2+m^2]|y\rangle. \quad (5)$$

In (4) we encounter the difference $\gamma_\mu S\gamma_\mu - \gamma_\mu\gamma_5 S\gamma_\mu\gamma_5$. This projects out the part of $S$ that *commutes* with $\gamma_5$; I call it $E(x,y;m) = \langle x|E|y\rangle$, where

$$E = m/[-(\not{D})^2+m^2]. \quad (6)$$

Now, $E$ is a positive definite operator; that is, $\langle\lambda|E|\lambda\rangle > 0$ for any state $|\lambda\rangle$. This observation is the key to the present argument.

We now have

$$\langle V_\mu^3(k)V_\mu^3(-k)\rangle - \langle A_\mu^3(k)A_\mu^3(-k)\rangle = (4/V)\int d\mu \int d^4x\, d^4y\, e^{ik\cdot(x-y)}\,\mathrm{Tr}\,\gamma_\mu E(x,y;m)\gamma_\mu E(y,x;m). \quad (7)$$

We define an operator $M_\mu(k)$ that consists of multiplication $\gamma_\mu e^{ik\cdot x}$. Specifically $[M_\mu(k)\psi](x) = \gamma_\mu\psi(x)e^{ik\cdot x}$ for any $\psi$. The adjoint operator $M_\mu{}^*(k)$ consists of multiplication by $\gamma_\mu e^{-ik\cdot x}$. Then (7) can be written

$$\langle V_\mu^3(k)V_\mu^3(-k)\rangle - \langle A_\mu^3(k)A_\mu^3(-k)\rangle = (4/V)\int d\mu\,\mathrm{Tr}M_\mu(k)EM_\mu{}^*(k)E, \quad (8)$$

where now the trace refers to a summation over spinor indices and an integral over $x$; summation over subscript $\mu$ is implied.

For any operator $M_\mu$ and any positive operator $E$, $\mathrm{Tr}M_\mu EM_\mu{}^*E$ is nonnegative. (Let $E = \sum_i \lambda_i |i\rangle \times\langle i|$ with $\lambda_i \geq 0$. Then $\mathrm{Tr}\,M_\mu EM_\mu{}^*E = \sum_{i,j}|\langle i|M_\mu \times|j\rangle|^2\lambda_i\lambda_j \geq 0$.) So (8) is positive. This completes the proof that the electromagnetic mass shift of the pion is positive.

The sign of this mass shift would have a particular significance if the bare masses of the up and down quarks were zero. In that case, the charged pions would be massless up to electromagnetic corrections. If these corrections were negative, the charged pions would become tachyons, triggering a shift in the vacuum and spontaneous breakdown of electromagnetic gauge invariance.

Just this question arises in technicolor theories,[10,11] in which one typically considers fermions of zero bare mass with superstrong gauge interactions. The phenomenology of these theories depends crucially[10] on the outcome of the vacuum alignment problem, which in turn depends on the sign of the analog of the electromagnetic mass shift of the charged pions. The present argument, by its nature, clearly applies to arbitrary vectorlike technicolor theories, settling the vacuum alignment problem in agreement with previous arguments[5,8] of a less rigorous nature. Insofar as the arguments of Refs. 5 and 8 are valid, the present argument also indicates that the analog of the $\rho$ meson is always lighter than the analog of the $A_\perp$ meson in vectorlike theories.

The second inequality to be presented here is a refinement of considerations of Ref. 2 and has been considered independently by Nussinov. Let $\alpha$ and $\beta$ be two quark flavors. Of all mesons with $\alpha\bar{\beta}$ quantum numbers, the lightest is a pseudoscalar.[2] The two-point function in the pseudoscalar channel is

$$\langle \bar{\alpha}i\gamma_5\beta(x)\bar{\beta}i\gamma_5\alpha(0)\rangle = \int d\mu\,\mathrm{Tr}\,\gamma_5 S(x,0;m_\alpha)^A\gamma_5 S(0,x;m_\beta)^A. \quad (9)$$

Let $U = S(x,0;m_\alpha)^A$ and $V = S(x,0;m_\beta)^A$. Since[2] $S(0,x;m)^A = \gamma_5 S^*(x,0;m)\gamma_5$ (where $S^*$ is the adjoint of $S$, regarded as a matrix in spinor space), we have

$$\langle\bar\alpha i\gamma_5\beta(x)\bar\beta i\gamma_5\alpha(0)\rangle=\int d\mu\,\mathrm{Tr}UV^*. \quad (10)$$

Now, suppose that in the $\alpha\bar\alpha$ or $\beta\bar\beta$ channels, gluon intermediate states can be neglected (Fig. 1). This is true for large $N$ for any $\alpha$ or $\beta$. It is true for any $N$ if $m_\alpha$ or $m_\beta$ is large. It is true in weakly coupled vectorlike theories such as quantum electrodynamics. And it is true in any case if instead of $\alpha\bar\alpha$ we consider $\alpha'\bar\alpha$, where $\alpha'$ and $\alpha$ are two distinct quarks of essentially equal mass (for instance, the up and down quarks). Then the lightest $\alpha\bar\alpha$ or $\beta\bar\beta$ mesons are pseudoscalars,[2] and

$$\begin{aligned}\langle\bar\alpha i\gamma_5\alpha(x)\bar\alpha i\gamma_5\alpha(0)\rangle &=\int d\mu\,\mathrm{Tr}\,\gamma_5 S(x,0;m_\alpha)^A\gamma_5 S(0,x;m_\alpha)^A\\ &=\int d\mu\,\mathrm{Tr}UU^*,\\ \langle\bar\beta i\gamma_5\beta(x)\bar\beta i\gamma_5\beta(0)\rangle &=\int d\mu\,\mathrm{Tr}VV^*.\end{aligned} \quad (11)$$

By the Cauchy-Schwarz inequality as used in Ref. 2,

$$\left|\int d\mu\,\mathrm{Tr}UV^*\right|^2\le\int d\mu\,\mathrm{Tr}UU^*\int d\mu\,\mathrm{Tr}VV^*, \quad (12)$$

so that

$$|\langle\bar\alpha i\gamma_5\beta(x)\bar\beta i\gamma_5\alpha(0)\rangle|^2 \le\langle\bar\alpha i\gamma_5\alpha(x)\bar\alpha i\gamma_5\alpha(0)\rangle\langle\bar\beta i\gamma_5\beta(x)\bar\beta i\gamma_5\beta(0)\rangle. \quad (13)$$

If there is a mass gap in each channel, so that

$$\begin{aligned}\langle\bar\alpha i\gamma_5\beta(x)\bar\beta i\gamma_5\alpha(0)\rangle &\xrightarrow{|x|\to\infty}\exp(-m_{\alpha\bar\beta}|x|),\\ \langle\bar\alpha i\gamma_5\alpha(x)\bar\alpha i\gamma_5\alpha(0)\rangle &\xrightarrow{|x|\to\infty}\exp(-m_{\alpha\bar\alpha}|x|),\\ \langle\bar\beta i\gamma_5\beta(x)\bar\beta i\gamma_5\beta(0)\rangle &\xrightarrow{|x|\to\infty}\exp(-m_{\beta\bar\beta}|x|),\end{aligned} \quad (14)$$

then (13) means

$$2m_{\alpha\bar\beta}\ge m_{\alpha\bar\alpha}+m_{\beta\bar\beta}. \quad (15)$$

Before comparing to experimental data, let us check that (15) is reasonable by comparing to some theoretical formulas. For very weak coupling (in QED or in QCD with quarks weighing hundreds of gigaelectronvolts), the masses in (15) can be evaluated using a Rydberg formula:

$$\begin{aligned}m_{\alpha\bar\beta}&=m_\alpha+m_\beta-\frac{1}{2}\left(\frac{e^2}{hc}\right)^2\frac{m_\alpha m_\beta}{m_\alpha+m_\beta},\\ m_{\alpha\bar\alpha}&=2m_\alpha-\frac{1}{2}\left(\frac{e^2}{hc}\right)^2\frac{m_\alpha}{2},\\ m_{\beta\bar\beta}&=2m_\beta-\frac{1}{2}\left(\frac{e^2}{hc}\right)^2\frac{m_\beta}{2}.\end{aligned} \quad (16)$$

Then (15) amounts to the statement that

$$\frac{m_\alpha}{2}+\frac{m_\beta}{2}\ge\frac{2m_\alpha m_\beta}{m_\alpha+m_\beta} \quad (17)$$

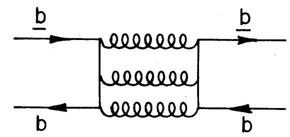


FIG. 1. $q\bar q$ annihilation into a gluon intermediate state is suppressed under certain conditions described in the text.

which may be readily verified.

For large $N$, (15) should hold for arbitrary values of $m_\alpha$ and $m_\beta$. If $m_\alpha$ and $m_\beta$ are small, the masses in (15) can be evaluated by current algebra; we expect

$$\begin{aligned}&m_{\alpha\bar\beta}=\lambda(m_\alpha+m_\beta)^{1/2},\quad m_{\alpha\bar\alpha}=\lambda(2m_\alpha)^{1/2},\\ &m_{\beta\bar\beta}=\lambda(2m_\beta)^{1/2},\end{aligned} \quad (18)$$

with a common constant $\lambda$. Then (15) amounts to the true statement that

$$2(m_\alpha+m_\beta)^{1/2}\ge(2m_\alpha)^{1/2}+(2m_\beta)^{1/2}. \quad (19)$$

Comparing now to experimental data, we first take $\alpha$ to refer to the up or down quark; we can take $m_{\alpha\bar\alpha}$ to refer to $m_{u\bar d}=m_\pi$. We take $\beta$ to be the charmed quark. Mass shifts due to gluon intermediate states are certainly tiny for $c\bar c$ states, in view of the very small widths of these states. So we take the $\eta_c$ as the lightest $c\bar c$ state, and the $D$ as the lightest $c\bar u$ or $c\bar d$ state, and we expect from (15) that

$$2m_D\ge m_\pi+m_{\eta_c}. \quad (20)$$

This is certainly valid, with $m_D=1870$ MeV, $m_\pi=135$ MeV, and $m_{\eta_c}=2980$ MeV. [Equation (20) is somewhat ambiguous, since the pion mass would be rather different if the up and down quarks were degenerate, but the ambiguity is well within the margin by which (20) is satisfied.]

Now we take $\alpha$ to be the strange quark and $\beta$ the charmed quark. Unfortunately, for $s\bar s$ pseudoscalars it is not a good approximation to ignore annihilation into gluons, as the $\eta$ and $\eta'$ masses show. However, with some loss in rigor we can compare (15) to experimental data as follows. We use current algebra to estimate that in the absence of gluon annihilation the masses of an $s\bar s$ pseudoscalar would have been

$$\begin{aligned}m_{s\bar s}&=[(m_{s\bar u})^2+(m_{s\bar d})^2-(m_{u\bar d})^2]^{1/2}\\ &=(m_{K^+}{}^2+m_{K^0}{}^2-m_\pi{}^2)^{1/2}.\end{aligned} \quad (21)$$

For $m_{c\bar s}$ and $m_{c\bar c}$ we take $m_F$ and $m_{\eta_c}$, respectively, so that the inequality (15) becomes

$$2m_F\ge(m_{K^+}{}^2+m_{K^0}{}^2-m_\pi{}^2)^{1/2}+m_{\eta_c}, \quad (22)$$

which agrees with experiment (the left-hand side is 3.94 GeV and the right-hand side is 3.67 GeV).

If in the future the masses of $b\bar{c}$ and $b\bar{b}$ pseudoscalars are measured, (15) gives the clean prediction

$$m_{b\bar{c}} \geq \tfrac{1}{2}(m_{\eta_c} + m_{\eta_b}) \qquad (23)$$

which will be interesting to test.

It should be clear from the discussion that the proof that $2m_{\alpha\bar{\beta}} \geq m_{\alpha\bar{\alpha}} + m_{\beta\bar{\beta}}$ holds only for the lightest pseudoscalar in each channel. Nussinov,[6] however, has suggested the same inequality independently and has argued that it holds not just for pseudoscalars but for the lowest state in each partial wave channel. He has given a variational argument which, while not a complete proof, makes the result highly plausible. Nussinov has also suggested that the present inequality $m_{\pi^+} \geq m_{\pi^0}$ follows from the inequality $2m_{\alpha\bar{\beta}} \geq m_{\alpha\bar{\alpha}} + m_{\beta\bar{\beta}}$ in the full-fledged $SU(3) \otimes U(1)$ gauge theory of strong and electromagnetic interactions, with $\alpha$ and $\beta$ taken as up and down quarks.

In conclusion, let us discuss briefly what sort of regularization is compatible with the arguments given here. The basic property of QCD that has been used is that $\det(\not{D} + M) > 0$. To preserve this property, one might choose to define this determinant in the continuum for fixed $A_\mu$, and then do the $A_\mu$ integral with a suitable cutoff. A suitable cutoff would be the gauge-invariant Pauli-Villars treatment that was put on a rigorous mathematical basis in Ref. 9. This appears satisfactory. One encounters difficulties if one tries instead a lattice regularization. Kogut-Susskind fermions preserve the positivity of the fermion determinant; unfortunately, they do not respect all of the vector symmetries, limiting the class of questions that can be conveniently asked of the lattice theory. Wilson fermions, on the other hand, do not have a positive definite determinant, except (presumably) in the continuum limit.[12]

I would like to thank S. Nussinov for discussion of his results, and to thank C. Vafa for continuing discussions of rigorous QCD inequalities. This work was supported in part by National Science Foundation Grant No. PHY80-19754.

[1] E. T. Tomboulis, Phys. Rev. Lett. 50, 885 (1983).
[2] D. Weingarten, Phys. Rev. Lett. 51, 1830 (1983).
[3] G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum, New York, 1980).
[4] C. Vafa and E. Witten, to be published.
[5] M. E. Peskin, Nucl. Phys. B175, 197 (1980).
[6] S. Nussinov, Phys. Rev. Lett. 51, 2081 (1983).
[7] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Lett. 18, 759 (1967).
[8] J. P. Preskill, Nucl. Phys. B177, 21 (1981).
[9] M. Asorey and P. K. Mitter, Commun. Math. Phys. 80, 43 (1981).
[10] S. Weinberg, Phys. Rev. D 13, 974 (1976), and 19, 1277 (1979).
[11] L. Susskind, Phys. Rev. D 20, 2619 (1979).
[12] D. Weingarten, Phys. Lett. 109B, 57 (1982), and Nucl. Phys. B215 [FS7], 1 (1983); C. Bernard, T. Draper, and K. Olynyk, Phys. Rev. D 27, 223 (1983); R. Gupta and A. Patel, to be published; G. Pawley, D. Wallace, E. Marinari, and F. Rapuano, Nucl. Phys. B220 [FS8], 137 (1983); H. Lipps, G. Martinelli, R. Petronzio, and F. Rapuano, to be published.