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Percolation on Fractal Lattices

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Numerical evidence of a percolation phase transition on infinitely ramified exact fractals is presented. The percolation transition is studied by a real-space renormalization-group technique on a family of exact fractal lattices with fractal dimensionality $\bar{d}$ between 1 and 2. The critical exponents for percolation depend strongly on the geometry of the fractals.

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Phase transitions on fractal lattices[1-3] and the fractal properties[2,4-6] themselves have currently been a subject of intensive investigation. However, percolation, which is considered the most fundamental example of phase transition,[7] has not been studied yet on fractal lattices. This is because most research on phase transitions on fractals has been done on Sierpinski-gasket-like fractals which are finitely ramified.[5] On these fractals percolation takes place only in the uninteresting limit $p_c = 1$.

In this Letter we present a family of exact fractals with infinite ramification. Percolation on these fractals is nontrivial.[8] We present numerical evidence for a percolation phase transition as well as a simple real-space renormalization-group (RSRG) approach for calculating the percolation threshold p_c and some percolation exponents. The threshold p_c and the critical exponents are found to vary according to the fractal lattice specifications.

We first study the fractal shown in Fig. 1(a). This fractal has an infinite ramification since upon each iteration of Fig. 1(a) the number of connections between different parts of the fractal increases by a factor of 2. In fact, we find it useful to define the ramification exponent ρ as follows.[1] Suppose one can isolate a part of the fractal of linear size R by "cutting" it at the minimal number of places, $N(R)$. Then as a consequence of self-similarity

$$N(R) \approx R^{\rho}. \tag{1}$$

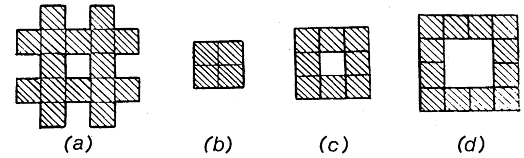

FIG. 1. (a) Fractal lattice used in Monte Carlo simulations. (b)–(d) Members of the fractal family (for b = 2, 3, and 4, respectively) studied by RSRG. In each case only the first iteration is shown.

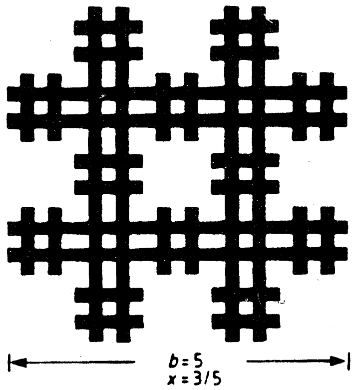

FIG. 2. The second iteration of Fig. 1(a). Each full row in Fig. 1(a) is split to two rows here; thus $\rho = \ln 2/\ln 5$.

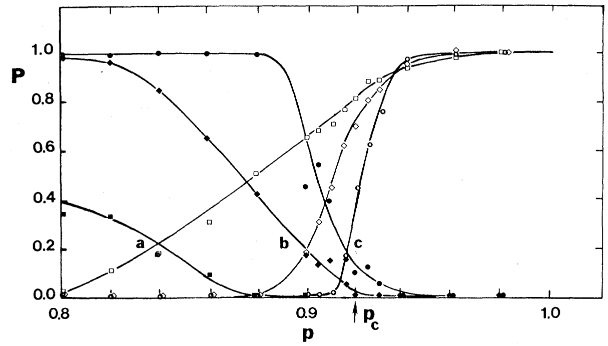

FIG. 3. Fraction of clusters reaching the edges of the lattice (white symbols) and of those which terminate before (black symbols), as a function of the concentration p for different lattice sizes: 25×25, square symbols; 125×125, diamonds; 625×625, circles.

The exponent ρ ranges between 0 for finitely ramified fractals (e.g., the Sierpinski gasket) to $\rho = d-1$ for homogeneous space. Moreover, let a fractal have a fractal dimensionality $\bar{d}$ and a resistivity exponent[5] $\bar{\zeta}$. Then in each section of the fractal of linear size R there are at least R^{ρ} fractal threads each being at most $R^{\bar{d}}/R^{\rho}$ long. Thus

$$R^{\bar{\zeta}} \leqslant R^{\bar{d}-2\rho} \quad \text{or} \quad \bar{\zeta} \leqslant \bar{d} - 2\rho. \tag{2}$$

It can be shown that $2-\bar{d} \leqslant \bar{\zeta}$ so that Eq. (2) gives a better bound than $\rho < d-1$.

We expect percolation to take place at $p_c < 1$ whenever $\rho > 0$. The fractal displayed in Fig. 1(a) (with $\bar{d} = \ln 16/\ln 5$) has $\rho = \ln 2/\ln 5$ so that nontrivial percolation may be expected. We carried out Monte Carlo simulations of site percolation on fractal lattices built from two, three, and four iterations of Fig. 1(a). The second iteration of Fig. 1(a) is shown in Fig. 2. On each of these exact fractal lattices we grew clusters by the cluster-growth method,[9] that is, a site near the center of the fractal was chosen as an origin and each site around it belonging to the fractal was designated as being occupied with probability p or not with probability $1-p$. The cluster growth was continued from the new cluster sites in a similar way till it either terminated or reached all the edges of the fractal. In Fig. 3 we show for each of the fractal lattices described above (25×25, 125×125, and 625×625) the fraction of clusters reaching all the edges as well as the fraction of those which terminate before reaching any of the edges,[10] as a function of the concentration p. The two curves intersect at points a, b, and c as shown in the figure. A sharper transition takes place as the lattice size increases. From the data for the largest lattice, $p_c = 0.915 \pm 0.010$. Also extrapolation of the intersection points a, b, and c yields a percolation threshold $p_c \simeq 0.92$. We apply to this problem a RSRG approach of Reynolds *et al.*[11] A renormalized *fractal* cell is said to be occupied with probability p' if there exists a percolating cluster from the lower edge of the fractal to the upper one. Thus,

$$p' = p^{12} + 12p^{11}(1-p) + 51p^{10}(1-p)^2 + 96p^9(1-p)^3 + 86p^8(1-p)^4 + 46p^7(1-p)^5 + 14p^6(1-p)^6 + 2p^5(1-p)^7, \tag{3}$$

which has the trivial fixed points $p^* = 0, 1$ but also the nontrivial one $p_c \simeq 0.9221$ in good agreement with the numerical data.

In order to investigate percolation as a function of the exponents $\bar{d}$ and ρ of the exact fractal lattice we present the following fractal family. The first iteration of each member of the family is based on a ring embedded in a square of $b\times b$ sites. In Figs. 1(b), 1(c), and 1(d) we show the first iteration of the cases of $b=2$, 3, and 4, respectively. Note that the case of $b=2$ is just a homogeneous square lattice. The fractal dimensionality of a member of size b is

$$\bar{d} = \ln[4(b-1)]/\ln b, \tag{4}$$

and ρ is

$$\rho = \ln 2/\ln b. \tag{5}$$

Thus, if we vary b, $\bar{d}$ ranges between 1 and 2 and ρ between 0 and 1. If one applies the same RSRG technique as before one obtains the general recursion formula

$$p' = 2p^b - p^{2b} \tag{6}$$

from which the percolation threshold p_c can be calculated. The correlation-length exponent is

$$\nu = \ln b/\ln[2b(1-p_c^{\,b-1})]. \tag{7}$$

The calculation of the exponent β is carried out by the ghost-site method[11] (the ghost site is attached to each site on the cluster with a probability h). The bond between the ghost site and an occupied cell renormalizes to $h'p'$ if there is a percolating path connecting the lower edge of the cell and the ghost site. Thus one gets for $b=2$ (homogeneous space)

$$h'p' = p^4 h_4 + 4p^3(1-p)h_3 + 2p^2(1-p)^2 h_2; \tag{8}$$

for $b=3$,

$$h'p' = p^8 h_8 + 8p^7(1-p)h_7 + p^6(1-p)^2(14h_6 + 3h_5 + 2h_3) + p^5(1-p)^3(10h_5 + 7h_4 + 2h_3) + p^4(1-p)^4(6h_4 + 4h_3) + 2p^3(1-p)^5 h_3; \tag{9}$$

and for $b = M \gg 1$,

$$h'p' = p^{4M-4}h_{4M-4} + (4M-4)p^{4M-5}(1-p)h_{4M-5} + \dots, \tag{10}$$

where $h_l = 1-(1-h)^l$. In Eq. (10) the missing terms make a negligible contribution. The exponent β is extracted by linearizing these equations at the critical point p_c [obtained from Eq. (6)] and $h_c = 0$. Then

$$\partial h'/\partial h|_{p_c,h_c} = d - \beta/\nu$$

so that using Eq. (4) and the value of ν found from Eq. (7) one is able to calculate β. Results for several values of b as well as for the limit $b = M \gg 1$ are displayed in Table I. According to this table p_c increases as $\bar{d}$ and ρ decrease towards the result obtained in one-dimensional space. The changes of ρ and $\bar{d}$ also have a dramatic effect on the exponents ν and β of the percolation. Note that the results obtained for the fractal in Fig. 1(a) are similar to those obtained in Table I for $b=4$.

TABLE I. Characteristic exponents of the fractal lattices and the critical exponents of percolation on these lattices.

Fractal			Percolation		
b	$\bar{d}$	ρ	p_c	ν	β
2	2	1	0.6	1.63	0.63
3	1.89	0.63	0.85	2.13	0.27
4	1.79	0.5	0.92	2.43	
5	1.72	0.43	0.95	2.69	
10	1.56	0.30	0.989	3.60	
100	1.30	0.15	0.999 90	6.79	
$M \gg 1$	$1+\frac{\ln 4}{\ln M}$	$\frac{\ln 2}{\ln M}$	$1-\frac{1}{M^2}$	$\frac{\ln M}{\ln 2}$	$\frac{\ln M}{M^2 \ln 2}$

An important result of this work is the presentation of a physical model to the problem of the dimensionality d approaching to unity from above.[12,13] Table I implies that for $\bar{d} \to 1$ ($M \gg 1$) the critical exponents are $\nu \approx 2/(\bar{d}-1)$, $\beta \sim [2/(\bar{d}-1)]\exp[-4\ln 2/(\bar{d}-1)]$, and $p_c \sim 1-\exp[-4\ln 2/(\bar{d}-1)]$. It is interesting to note that these results have the same dependence on $d-1$ as those obtained by the renormalization-group technique.[12,13] The constants, however, are different because of the specific geometric structure of the fractal family used.

We have shown that a percolation transition takes place on exact fractal lattices with infinite ramification. We have suggested the exponent ρ to characterize the ramification and given bounds on it. The question of whether ρ is derivable from other critical exponents or not remains open. Finally, we note that an interesting situation arises for fractals with a fracton dimensionality ranging above and below $\bar{\bar{d}}_f = \frac{4}{3}$ because of a conjecture by Alexander and Orbach.[4] It is reasonable that $\bar{\bar{d}}_f$ of the fractal should be bigger than or equal to $\bar{\bar{d}}_p$ of the percolation cluster on the fractal. Then for $\bar{\bar{d}}_f$ higher than $\frac{4}{3}$ one expects the conjecture for percolation that $\bar{\bar{d}}_p = \frac{4}{3}$ to hold. It would be interesting to have this conjecture checked for percolation on fractal lattices. Also the question what happens for $\bar{\bar{d}}_f$ less than $\frac{4}{3}$ is still unanswered. What would be the required value of $\bar{\bar{d}}_p$? There is still much numerical and theoretical work to be done on this intriguing subject.

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