Dynamic Stability of a Doubly Diffusive System under Parametric Modulation

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The effect of modulating the temperature difference between the plates at the onset of convection in a doubly diffusive system has been studied. Both Stern and Veronis configurations show stabilization and destabilization depending on the modulation frequency, in contrast with the behavior of single-component systems. The stability effects are particularly pronounced in the Veronis configuration because of parametric resonance.

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The response of a dynamical system to a periodically modulated driving force[1,2] is receiving much attention in the hydrodynamic context.[3,4] For the Rayleigh-Bénard problem the effect of periodic modulation of the temperature difference between the plates is known to provide dynamic stabilization and delay the onset of convection if finite-size effects are ignored.[5] The onset of convective flow in double-diffusive systems, which we study below, exhibits a far richer stability pattern.[6] The possibility of the onset of an oscillatory flow in the unmodulated system allows for parametric resonance in the modulated one and generally leads to an enhanced response of the system. All the features to be discussed below should be amenable to experimental verification.

A double-diffusive system is characterized by two different diffusivities—usually those of heat and the solute. The difference in diffusivities can drive convection in the system even when it is hydrostatically stable in the single-component sense. The configuration where the temperature and concentration (of the solute) both increase in the upward direction is the Stern configuration,[7] or the "fingering" regime, whereas the reverse situation is known as the Veronis configuration,[8] or the "diffusive" regime. The instability is stationary in the Stern configuration and nearly always oscillatory in the Veronis configuration.

To study the effect of periodic modulation in the double-diffusive system, we assume that the concentration gradient is fixed while the temperature of one of the plates is modulated with frequency ω. Here we use the Lorenz-model-like truncation of the full hydrodynamic equations proposed by Veronis[8] and studied in detail by DaCosta, Knobler, and Weiss.[9] The use of the full hydrodynamic equations leads to very similar answers.[10] Our stability analysis is linear.

The truncation involves the following five Fourier components: (i) X, the 101 component of the velocity field in the vertical direction; (ii) Y, the 101 component of the temperature field; (iii) Z, the 002 component of the temperature field; (iv) U, the 101 component of the concentration field; (v) V, the 002 component of the concentration field.

The resulting differential equations can be written in the form

$$\dot{X}=\sigma(-X+Y+U), \tag{1a}$$

$$\dot{Y}=-XZ+r_1X-Y, \tag{1b}$$

$$\dot{Z}=XY-bZ, \tag{1c}$$

$$\dot{U}=XV-sr_2X-Us, \tag{1d}$$

$$\dot{V}=UX-sbV. \tag{1e}$$

Here $\sigma=\nu/D$ is the Prandtl number, $s=D_s/D$, $r_1=\alpha g(\Delta T)d^3/D\nu R_c$, $r_2=\beta g(\Delta T)d^3/D_s\nu R_c$. ν is the kinematic viscosity, D is the heat diffusion, D_s is the solutal diffusivity, $\alpha=-(1/\rho)\partial\rho/\partial T$ is the thermal expansion coefficient, $\beta=(1/\rho)\partial\rho/\partial C$, d is the separation of the plates, $R_c=27\pi^4/4$ is the critical Rayleigh number for free boundaries, the number b ranges between 1 and 4, and time is measured in units of $2d^2/3\pi^2D$.

It can be readily verified from Eqs. (1a)–(1e) by a linear stability analysis that the state of rest ($X=Y=Z=U=V=0$) is destabilized to the following: (i) a stationary state if

$$r_1-r_2=1; \tag{2}$$

(ii) an oscillatory state, with frequency

$$\omega_0^{\,2}=s+\sigma+s\sigma-r_1\sigma+r_2s\sigma=\frac{s\sigma(1-r_1+r_2)}{1+s+\sigma}, \tag{3}$$

if

$$\frac{r_1\sigma}{(1+s)(\sigma+s)}-\frac{r_2\sigma s}{(\sigma+1)(1+s)}=1. \tag{4}$$

Note that $r_1>0$ if the system is heated from below, and $r_2>0$ if the concentration decreases upwards. Keeping this in mind, we see that in the

Stern configuration ($r_1 < 0$, $r_2 < 0$) the instability is stationary, while in the Veronis configuration ($r_1 > 0$, $r_2 > 0$) it is nearly always oscillatory.

We now study the effect of modulating the temperature difference between the plates while holding the concentration difference fixed. This is equivalent to a frequency modulation on the parameter r_1 in Eqs. (1a)–(1e). The modified forms of Eqs. (1a)–(1e) are obtained by replacing r_1 by $\tilde{r}_1(1+\epsilon\cos\omega t)$ in Eq. (1b).[11] In the following we will assume that $\epsilon \ll 1$ and carry out a perturbative analysis to find the change in $\tilde{r}_1$ up to $O(\epsilon^2)$ from its unperturbed value. For this purpose we expand each mode A ($A = X, Y, Z, U, V$) as

$$A = A_0 + A_1\epsilon + A_2\epsilon^2 + \ldots \tag{5}$$

and the Rayleigh number as

$$\tilde{r}_1 = r_1^{(0)} + \epsilon r_1^{(1)} + \epsilon^2 r_1^{(2)} + \ldots . \tag{6}$$

Clearly $r_1^{(1)} = 0$ as the correction to the Rayleigh number cannot depend on the sign of ϵ. Inserting the above expansions in the linearized versions of Eqs. (1a)–(1e) and equating coefficients of like powers of ϵ, we obtain

$$L\begin{pmatrix} X_0 \\ Y_0 \\ U_0 \end{pmatrix} = 0, \tag{7a}$$

$$L\begin{pmatrix} X_1 \\ Y_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} 0 \\ r_1^{(0)} X_0 \cos\omega t \\ 0 \end{pmatrix}, \tag{7b}$$

$$L\begin{pmatrix} X_2 \\ Y_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ r_1^{(2)} X_0 + r_1^{(0)} X_1 \cos\omega t \\ 0 \end{pmatrix}, \tag{7c}$$

where L is the matrix

$$L = \begin{pmatrix} \partial/\partial t + \sigma & -\sigma & -\sigma \\ -r_1^{(0)} & \partial/\partial t + 1 & 0 \\ s r_2 & 0 & \partial/\partial t + s \end{pmatrix}. \tag{8}$$

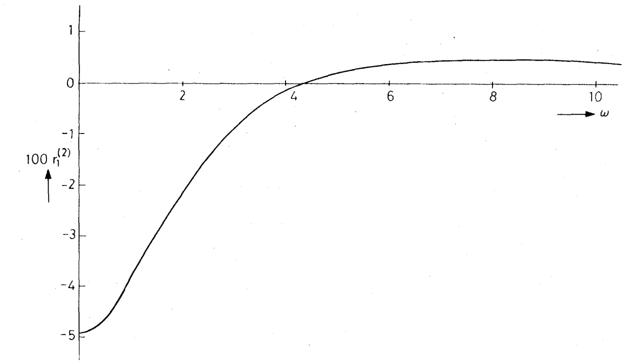

FIG. 1. Leading correction $r_1^{(2)}$ to the thermal Rayleigh number as a function of the modulating frequency ω for the stationary instability (Stern configuration). Negative and positive values of $r_1^{(2)}$ imply dynamic destabilization and stabilization, respectively.

We now discuss separately the Stern and Veronis configurations.

(1) *Stern configuration.*—In this case the instability is stationary (X_0, Y_0, and U_0 are time independent). The corrections X_1, Y_1, and U_1 due to modulation are time dependent and can be obtained from Eq. (7b) as

$$X_1 = X_0 \sigma r_1^{(0)} \frac{sL_1 + \omega L_2}{L_1^2 + L_2^2} \cos\omega t, \tag{9}$$

where

$$L_1 = -\omega^2(1 + S + \sigma) \tag{10}$$

and

$$L_2 = -\omega^2 + s + \sigma + s\sigma - \sigma(r_1^{(0)} - r_2 s). \tag{11}$$

We have used $r_1^{(0)} - r_2 = 1$. Turning now to Eq. (7c), we note that for consistency the time-averaged part of the right-hand side has to be orthogonal to the solution of Eq. (7a), and hence

$$r_1^{(2)} = -r_1^{(0)} \frac{\langle X_1 \cos\omega t\rangle}{X_0} = \sigma \frac{(r_1^{(0)})^2}{2} \frac{s(1+s+\sigma) - [s + \sigma + s\sigma - \sigma(r_1^{(0)} - r_2 s) - \omega^2]}{\omega^2[1+s+\sigma]^2 + [\sigma + s + s\sigma - \sigma(r_1^{(0)} - r_2 s) - \omega^2]^2}. \tag{12}$$

In Fig. 1, we show a plot of $r_1^{(2)}$ vs ω for $\sigma = 10$, $s = 10^{-2}$ (solutal diffusivity is usually much smaller than thermal diffusivity), $r_1^{(0)} = -1$, and $r_2 = -2$. Note that $r_1^{(2)} < 0$ for $\omega < (\sigma|r_2|)^{1/2}$ (approximately). This implies that in this frequency range the upper plate has to be taken to a higher temperature for convection to occur. This is in effect a destabilization because the stability of the system is supposed to increase as the temperature of the upper plate is increased.

(2) *Veronis configuration.*—Here we consider the case where the instability is oscillatory and assuming that $X_0 = \mathrm{Re}A_0 \exp(i\omega_0 t)$, we obtain from Eq. (7b) the particular integral

$$X_1(t) = \mathrm{Re}\left(\frac{\sigma r_1^{(0)}}{2} A_0 \frac{i(\omega_0+\omega)+s}{L_1 + iL_2} \exp[i(\omega_0+\omega)t] + (\omega \to -\omega)\right), \tag{13}$$

where

$$\bar{L}_1 = (1+\sigma+s)[\omega_0^2 - (\omega+\omega_0)^2], \quad \bar{L}_2 = (\omega_0+\omega)[\omega_0^2 - (\omega_0+\omega)^2]. \tag{14}$$

The above solution is valid for ω away from the parametric resonance at $\omega = 2\omega_0$.

At $\omega = 2\omega_0$, we cannot set $r_1^{(1)} = 0$ *a priori* and Eq. (7b) needs to be written as

$$L\begin{pmatrix} X_1 \\ Y_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} 0 \\ r_1^{(1)} X_0 + r_1^{(0)} X_0 \cos\omega t \\ 0 \end{pmatrix}. \tag{15}$$

With $X_0 = A_0 \cos\omega_0 t$, the consistency condition now yields

$$r_1^{(1)} = -\tfrac{1}{2} r_1^{(0)}. \tag{16}$$

Inserting this value of $r_1^{(1)}$ in Eq. (15), we obtain

$$X_1 = \mathrm{Re}\left(\frac{\sigma r_1^{(0)}}{2} A_0 \frac{s+3i\omega_0}{L_1' + iL_2'} \exp(3i\omega_0 t)\right), \tag{17}$$

with $L_1' = -8\omega_0^2(1+\sigma+s)$ and $L_2' = -24\omega_0^3$.

The second-order correction $r_1^{(2)}$ can be obtained from Eq. (7c) (for the case $\omega = 2\omega_0$, $r_1^{(1)}$ terms have to be included here) by imposing the consistency condition. We find

$$r_1^{(2)} = \begin{cases} \dfrac{\sigma(r_1^{(0)})^2}{4} \dfrac{(\omega-\omega_0)^2 + s(1+\sigma+s)}{[(1+\sigma+s)^2 + (\omega-\omega_0)^2][(\omega-\omega_0)^2 - \omega_0^2]} + (\omega \to -\omega) & \text{for } |\omega - 2\omega_0| \gg |\epsilon\omega_0|, \\ \dfrac{\sigma(r_1^{(0)})^2}{32\omega_0^2} \dfrac{9\omega_0^2 + s(1+\sigma+s)}{(1+\sigma+s)^2 + 9\omega_0^2} + \dfrac{r_1^{(0)}}{4} & \text{for } |\omega - 2\omega_0| = O(\epsilon\omega_0). \end{cases} \tag{18}$$

The above effects could be experimentally tested on thermohaline solutions. The fluid mixtures should show similar qualitative behavior and may be the preferable system for experimental purposes.

[1]N. Minorsky, *Nonlinear Oscillations* (Krieger, Huntington, N.Y., 1974).
[2]M. Lucke and Y. Saito, to be published.
[3]S. H. Davies, Annu. Rev. Fluid Mech. 8, 57 (1976).
[4]G. Venezian, J. Fluid Mech. 35, 243 (1969).
[5]G. Ahlers, P. C. Hohenberg, and M. Lucke, Bull. Am. Phys. Soc. 26, 1271 (1981).
[6]A similar comment was made by J. W. Rauscher and R. E. Kelly, Int. J. Heat Mass Transfer 18, 1216 (1975).
[7]M. E. Stern, Tellus 12, 172 (1960).
[8]G. Veronis, J. Mar. Res. 23, 1 (1965).
[9]L. N. DaCosta, E. Knobler, and N. O. Weiss, J. Fluid Mech. 109, 25 (1981).
[10]K. Banerjee and J. K. Bhattacharjee, to be published.
[11]In making this replacement, we are ignoring the dependence of the basic temperature in the fluid upon the frequency. At high frequencies the error is small; at moderate frequencies our results are expected to be qualitatively correct.