Linking Numbers, Spin, and Statistics of Solitons

Frank Wilczek
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106

and

A. Zee
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, and Department of Physics,[a] University of Washington, Seattle, Washington 98195

(Received 24 October 1983)

The spin and statistics of solitons in the (2 + 1)- and (3 + 1)-dimensional nonlinear σ models is considered. For the (2 + 1)-dimensional case, there is the possibility of fractional spin and exotic statistics; for 3 + 1 dimensions the usual spin-statistics relation is demonstrated. The linking-number interpretation of the Hopf invariant and the use of suspension considerably simplify the analysis.

PACS numbers: 02.40.+m, 11.10.Lm, 11.30.-j, 75.70.-i

The existence of solitons in the (2 + 1)-dimensional O(3) nonlinear σ model, as first discussed by Belavin and Polyakov,[1] is implied by the homotopy $\pi_2(s_2) = z$. The model is described by the functional

$$E = \int d^2x (1/2f)(\partial_i n^a)^2, \quad i = 1, 2; \quad a = 1, 2, 3, \tag{1}$$

giving the energy of a static configuration specified by $n^a(\vec{x})$. The "order parameter" n^a is a three-dimensional unit vector: $n^a n^a = 1$. If we describe the ground state by $\vec{n}(\vec{x}) = (0, 0, 1)$, then the basic soliton is described by

$$\vec{n}(\vec{x}) = (\hat{x} \sin f, \cos f). \tag{2}$$

Here $\hat{x} = \vec{x}/|\vec{x}|$ denotes the two-dimensional unit radial vector and $f(r = |\vec{x}|)$ is a function varying smoothly and monotonically from $f(0) = \pi$ to $f(\infty) = 0$ as r increases. We refer to such a topological configuration as a skyrmion.[2] The topological current in this model is

$$J^\mu = (1/8\pi)\epsilon^{\mu\nu\lambda}\epsilon^{abc} n^a \partial_\nu n^b \partial_\lambda n^c. \tag{3}$$

The space-time indices $\mu, \nu, \ldots$ run over 0, 1, 2. One easily verifies the conservation of this. The topological charge of this current,

$$Q = \int d^2x\, J^0 = (1/8\pi)\int d^2x\, \epsilon^{ij}\epsilon^{abc} n^a \partial_i n^b \partial_j n^c, \tag{4}$$

clearly describes the homotopy of the mapping $s_2 \to s_2$ for $\vec{n}$ satisfying the boundary condition $\vec{n}(\vec{x} = \infty) = \text{const}$. The skyrmion has $Q = 1$. By using Bogomolny's inequality, one can solve exactly the problem of minimizing the energy functional for a given Q. Finally, we mention that this model[3] provides a phenomenological description of Heisenberg ferromagnets in a two-dimensional system and thus the phenomena exhibited in this model may conceivably be accessible experimentally.

In this paper we point out that the skyrmion may possess fractional angular momentum and obey peculiar quantum statistics. One of us had previously proposed[4-6] the possibility of fractional angular momentum and of statistics which are neither Bose-Einstein nor Fermi-Dirac. As we will see, the (2 + 1)-dimensional O(3) nonlinear σ model provides an amusing and explicit field-theoretic realization of these ideas. Our discussion is also related to several other field-theoretic phenomena discovered in recent years.

The relevant mathematics which allows skyrmions to have these peculiar properties is the homotopy $\pi_3(s_2) = z$ [which is perhaps somewhat less obvious than the homotopy $\pi_2(s_2) = z$ responsible for the skyrmion's existence]. It is easy to exhibit the basic Hopf map of $s_3 \to s_2$. In fact, physicists should be familiar with this fact from elementary discussions of the Pauli matrices σ^a. Define $n^a = z^\dagger \sigma^a z$ where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

is a complex two-component spinor with the constraint $|z_1|^2 + |z_2|^2 = 1$. Notice that the U(1) transformation $z \to e^{i\theta} z$ leaves n invariant and so the inverse image of any point on s_2 is a circle on s_3. What is not so obvious is the construction of a Hopt invariant to describe $\pi_3(s_2)$ [just as Q describes $\pi_2(s_2)$]. This will be explained below.

Let us first address the physical question of the spin of the skyrmion. To determine the spin, we rotate the skyrmion adiabatically through 2π over a long time period T. According to Feynman, at the end of this rotation the wave function acquires a phase factor e^{iS} where S is the action corresponding to the adiabatic rotation. The angular momentum J of the skyrmion is given by $e^{iS} = e^{i2\pi J}$.

Now, if S has simply the standard form [cf. Eq. (1)]

$$S_0 = \int d^3x (1/2f)(\partial_\mu n^a)^2, \quad (5)$$

then it is easy to see that S_0 is of order $1/T \to 0$ as $T \to \infty$. The skyrmion has $J=0$. However, we have not taken into account the possibility of including in S a topological term. This possibility is by now familiar from the discussion of the θ vacua[7] in quantum chromodynamics, from studies of three-dimensional Yang-Mills theory and gravity,[8] and from recent work of Witten[9] on strongly interacting skyrmions[10] (based on earlier work of Wess and Zumino[11]). In general, we can have $S = S_0 + \theta H$ where θ is a real parameter and H is the Hopf invariant which we now define.

The conservation of J^μ licenses us to manufacture a "gauge potential" A^μ by the curl equation:

$$J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \equiv \tfrac{1}{2} \epsilon^{\mu\nu\lambda} F_{\nu\lambda}. \quad (6)$$

A_μ is defined up to the gauge freedom $A_\mu \to A_\mu - \partial_\mu \Lambda$. Note that A_μ depends nonlocally on $n^a(x)$. In the gauge $\partial A = 0$, we have $A_\mu = -\partial^{-2} \epsilon_{\mu\nu\lambda} \partial_\nu J_\lambda$. [An alternative construction is to write $A_\mu = iz^\dagger \partial_\mu z$. The U(1) phase rotation on z induces the gauge transformation on A_μ.] The Hopt invariant is defined by

$$H = -(1/4\pi) \int d^3x\, \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}$$
$$= -(1/2\pi) \int d^3x\, A_\mu J^\mu. \quad (7)$$

H is obviously invariant under gauge transformation on A_μ. [We note that this is just the Abelian version of the topological term studied by Deser, Jackiw, and Templeton,[8] but since H is gauge invariant, θ is not quantized. Furthermore, here H is to be regarded as a functional of $\vec{n}(\vec{x})$. In the language of Zumino, Wu, and Zee,[12] H is proportional to $\int \omega_3{}^0$. If $\hat{\mu}$ denotes a four-dimensional index then $\partial_{\hat{\rho}} \epsilon^{\hat{\rho}\hat{\mu}\hat{\nu}\hat{\lambda}} A_{\hat{\mu}} F_{\hat{\nu}\hat{\lambda}} = \tfrac{1}{2} \epsilon^{\hat{\rho}\hat{\mu}\hat{\nu}\hat{\lambda}} F_{\hat{\rho}\hat{\mu}} F_{\hat{\nu}\hat{\lambda}}$, connecting the Hopf invariant to the chiral anomaly.]

Spatial rotation of a single skrymion is equivalent to an isospin rotation and thus we evaluate H for the time-varying configuration $n_1 \pm i n_3 = e^{\pm i\alpha(t)} \times (\hat{n}_1 \pm i\hat{n}_2)(\vec{x})$, $n_3 = \hat{n}_3(\vec{x})$. {Strictly speaking, this defines a map of $s_2 \times [0,1] \to s_2$.} It is not necessary to know the explicit form of n_a. From Eq. (3) we find

$$J_i = -(1/8\pi)(d\alpha/dt) \epsilon_{ij} \partial_j n_3. \quad (8)$$

It suffices to know that $J_0(r) = \epsilon_{ij} \partial_i A_j$ is a function of r to determine $A_j = -\epsilon_{jk} x_k g(r)/r^2$ where $g(r) = \int_0^r dr' r' J_0(r')$. This and Eq. (8) allow us to determine $A_0 = -(d\alpha/dt) n_3$. Inserting into Eq. (7) we find

$$H = g(\infty) n_3(\infty) [\alpha(T) - \alpha(0)]/2\pi = 1. \quad (9)$$

The skyrmion has angular momentum $\theta/2\pi$.

For a ferromagnet, θ should be determined by the microscopic theory underlying the phenomenological σ model.

It is easy to show that H is a homotopic invariant for $s_3 \to s_2$. Consider a map with $\vec{n}(\vec{x}, t=\infty)$ constant and a small deformation of $\vec{n}$ leaving invariant $\vec{n}(\infty)$. Then

$$\delta J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu \epsilon_{abc} 2n_a \delta n^b \partial_\lambda n^c$$
$$= \epsilon_{\mu\nu\lambda} \partial_\nu \delta A_\lambda \quad (10)$$

and we find

$$\delta H = (-1/2\pi) 2 \int d^3x\, \delta A_\mu J^\mu = 0.$$

There is a deep theorem[13] which equates the Hopf invariant to the linking number between two curves in R^3. To have a heuristic understanding of this, consider the maps $s_3 \to s_2$. The reverse image of a point in s_2 is a curve in s_3 which by a stereographic projection we can think of as a curve in R^3 (with ∞ identified as one point). Thus, for the basic map given explicitly above, $\vec{n} = (0, 0, 1)$ corresponds to the great circle $|z_1| = 1$, $z_2 = 0$ on s_3, while $\vec{n} = (0,0,-1)$ corresponds to $z_1 = 0$, $|z_2| = 1$. Write the real components of (z_1, z_2) as $(\cos\psi, \sin\psi\cos\theta, \sin\psi\sin\theta\cos\varphi, \sin\psi\sin\theta\sin\varphi)$ and stereographically project this point to $r(\psi) \times (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$ in R^3 where $r(\psi)$ ranges monotonically from ∞ to 0 as ψ ranges from 0 to π. We see that the curves corresponding to $\vec{n} = (0,0,1)$ and to $\vec{n} = (0,0,-1)$ link once. The reader may find it amusing to work out the curves corresponding to other points.

Using this linking theorem, we can easily determine the spin and statistics of a skyrmion. Consider the following process in 2+1 dimensions. At some time create a pair of skyrmion and antiskyrmion and pull them apart. Rotate the skyrmion through 2π. Allow the pair to come together. Since at ∞ we have the physical vacuum this defines a map $s_3 \to s_2$. Were the skyrmion not rotated, the map would be homotopically trivial. Here, the corresponding map has Hopt invariant 1. The two curves traced out by two specific values of $\vec{n}$ will be linked once as indicated in Fig. 1.

To determine the statistics obeyed by a skyrmion we consider a process in which we create two skyrmion-antiskyrmion pairs and subsequently

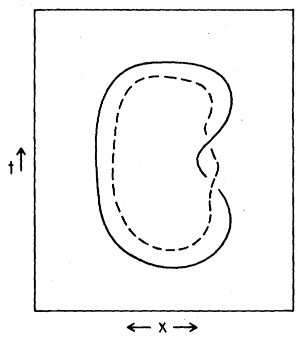

FIG. 1. The creation and annihilation of a skyrmion-antiskyrmion pair, with a 2π rotation of the skyrmion. The two curves correspond to $\vec{n}=(0,0,1)$ and $(1,0,0)$, say.

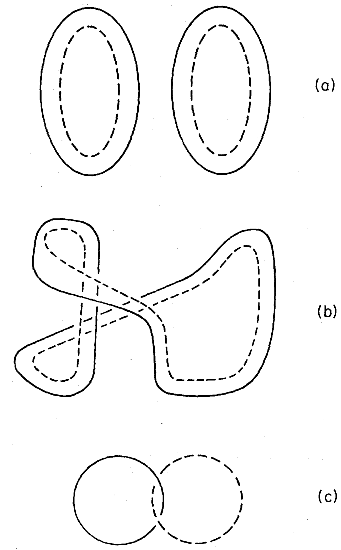

FIG. 2. (a) The creation and annihilation of two skyrmion-antiskyrmion pairs. (b) The process in (a) but with an interchange of the two skyrmions. (c) The two curves in (b) after a homotopic deformation.

bring them to annihilation but after interchanging the two skyrmions. We see, by the maneuvering indicated in Fig. 2, that the linking number is 1 for this process. The map of $s_3 \to s_2$ corresponding to this process therefore has Hopf invariant 1. Thus, the skyrmion obeys exotic statistics which interpolates continuously between Bose and Fermi statistics as described in Ref. 6. (By the way, the alternative of directly computing the Hopf integral corresponding to rotating a pair of skyrmions through π appears to be quite difficult.) Note that the discussion there is for a gauge theory. Here, we do not have a gauge theory but, curiously, one can manufacture a gauge potential A_μ.

Given a map $f: s_k \to s_n$ one can always construct[13] a map $\bar{f}: s_{k+1} \to s_{n+1}$ (called the Freudenthal suspension of f) by $\bar{f}(t, (1-t^2)^{1/2}x) = (t, (1-t^2)^{1/2}f(x))$ where $x \in s_k$ and $t \in [0,1]$. This induces a homomorphism[13] $F: \pi_k(s_n) \to \pi_{k+1}(s_{n+1})$ of the homotopy classes of f and $\bar{f}$. Our discussion can thus be "suspended" into (3+1)-dimensional space-time: $\pi_2(s_2) \to \pi_3(s_3) = z$ and $\pi_3(s_2) \to \pi_4(s_3) = z_2$. The first of these is an isomorphism, the second is onto: The suspension of a map $s^3 \to s^2$ to a map $s^4 \to s^3$ is nontrivial if and only if the map has odd Hopf invariant. Since $s_3 = \mathrm{SU}(2)$ manifold, the homotopy $\pi_3(s_3)$ implies the existence of skyrmions in the $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ nonlinear σ model. The fact that $\pi_4(s_3) = z_2$ allows one to quantize the skyrmion as a spin-$\frac{1}{2}$ fermion as discussed by Witten.[9] It is consistent with the standard three-space angular momentum analysis and with the well-known facts $\pi_1(\mathrm{SO}(2)) = z$ and $\pi_1(\mathrm{SO}(3)) = z_2$ that $\pi_4(s_3)$ is z_2 rather than z.

This material is based upon research supported in part by the National Science Foundation under Grant No. PHY77-27084, supplemented by funds from the National Aeronautics and Space Administration.

---

[a] On leave of absence 1983–1984.

[1] A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 22, 503 (1975) [JETP Lett. 22, 245 (1975)].

[2] T. H. R. Skyrme, Proc. Roy. Soc. London, Ser. A 247, 260 (1958).

[3] For a review of the material in this introductory paragraph, see R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[4] Some aspects of this subject appear to have been anticipated in the remarkable paper of D. Finkelstein and J. Rubinstein, J. Math. Phys. 9, 1762 (1968).

[5] P. Hasenfratz, Phys. Lett. 85B, 338 (1979); J. Schonfeld, Nucl. Phys. B185, 157 (1981).

[6] F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982), and 49, 957 (1982).

[7] G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

[8] J. Schonfeld, Ref. 5; S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. 48, 975 (1982), and Ann. Phys. (N.Y.) 140, 372 (1982); see also Y.-S. Wu and A. Zee, to be published.

[9] E. Witten, Nucl. Phys. B223, 422, 433 (1983).

[10] A. P. Balashandran, V. P. Nair, and C. G. Trahern, Phys. Rev. Lett. 49, 1124 (1982).

[11] J. Wess and B. Zumino, Phys. Lett. 37B, 95 (1971).

[12] B. Zumino, Y.-S. Wu, and A. Zee, to be published.

[13] For example, P. J. Hilton, *An Introduction to Homotopy Theory* (Cambridge Univ. Press, Cambridge, England, 1953), Chap. VI.