Diffeomorphism Groups, Gauge Groups, and Quantum Theory

Gerald A. Goldin
Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

and

R. Menikoff and D. H. Sharp
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 27 September 1983)

Unitary representations of the infinite-parameter group Diff(R^3) are presented which describe particles with spin as well as tightly bound composite particles. These results support the idea that Diff(R^3) can serve as a "universal group" for quantum theory.

PACS numbers: 02.20.+b, 03.65.Bk

This paper reports new findings concerning the physics of unitary representations of the group of diffeomorphisms of R^3 [Diff(R^3)]. We show that such representations can describe quantum particles with various kinds of internal structure, including particles with spin as well as composite particles having dipole, quadrupole, and higher multipole moments. We also present a heuristic description of representations based on string and loop configurations. Finally, we observe that the notion of a gauge group has a natural expression in this setting.

The infinite-parameter group Diff(R^3) arises in physics in at least three distinct contexts: fluid dynamics, general relativity, and quantum field theory. Consider first the configurations of a classical, compressible fluid. A diffeomorphism describes the displacement of such a fluid from a fixed initial configuration. Mathematically, it is a mapping from a domain Ω in R^3 to itself which is one-to-one, onto, continuous, continuously invertible, and infinitely differentiable. Such mappings form a group under the operation of composition. Thus the configuration space is the infinite-dimensional manifold of Diff(Ω). The tangent space to this manifold is the space of C^∞ vector fields on Ω (velocity fields); the phase space is the cotangent bundle $T^*(\mathrm{Diff}(\Omega))$. For incompressible fluids, one considers the group s Diff(Ω) of volume-preserving diffeomorphisms, and the tangent space of divergence-free velocity fields. The dynamics of such fluids can be studied by means of noncanonical Poisson brackets, satisfying the algebra of Lie brackets of vector fields (see below).[1] One is thus led to consider the quantization of such structures by means of unitary representations of the diffeomorphism group.

Secondly, in the "superspace" picture in general relativity one considers metrics g_{ij} on a spacelike three-dimensional manifold M, taking as equivalent those which are related by a diffeomorphism. The diffeomorphism group thus plays the role of a gauge group for the theory. In a canonical approach to the quantization of gravity, the state functionals on the space of three-metrics would carry a representation of Diff(M) intertwining the representation of the canonical commutation relations. Such representations could thus illuminate our understanding of quantum gravity.[2]

Third, consider a second-quantized nonrelativistic field $\psi(\vec{x},t)$ satisfying either canonical commutation (−) or anticommutation (+) relations $[\psi(\vec{x}),\psi^*(\vec{y})]_\pm=\delta(\vec{x}-\vec{y})$ at equal times. In the Fock representation of ψ, define the mass density $\rho(\vec{x}) = m\psi^*(\vec{x})\psi(\vec{x})$ and the momentum density

$$\vec{J}(\vec{x})=(\hbar/2i)\{\psi^*(\vec{x})[\nabla\psi(\vec{x})]-[\nabla\psi^*(\vec{x})]\psi(\vec{x})\}.$$

Consider the averaged operators $\rho(f)=\int\rho(\vec{x})f(\vec{x})d^3x$ and $J(\vec{g})=\int\vec{J}(\vec{x})\cdot\vec{g}(\vec{x})d^3x$, where f and the components of $\vec{g}$ are C^∞ functions of rapid decrease on R^3 (Schwartz's space functions). These operators satisfy the Lie algebra

$$[\rho(f_1),\rho(f_2)]=0, \tag{1}$$

$$[\rho(f),J(\vec{g})]=i\hbar\rho(\vec{g}\cdot\nabla f), \tag{2}$$

$$[J(\vec{g}_1),J(\vec{g}_2)]=i\hbar J([\vec{g}_1,\vec{g}_2]), \tag{3}$$

where $[\vec{g}_1,\vec{g}_2]=\vec{g}_2\cdot\nabla\vec{g}_1-\vec{g}_1\cdot\nabla\vec{g}_2$ is the Lie bracket of the vector fields $\vec{g}_1$ and $\vec{g}_2$. It is remarkable that the same Lie algebra (1)–(3) occurs whether one starts with commuting or anticommuting fields. The group obtained by exponentiating the current commutators is a semidirect product $\mathcal{S}\wedge\mathrm{Diff}(R^3)$, where $\mathcal{S}$ is Schwartz's space. For $\vec{\varphi}(\vec{x})\in\mathrm{Diff}(R^3)$, the condition $\vec{\varphi}(\vec{x})\to\vec{x}$ rapidly as $|\vec{x}|\to\infty$ is imposed which permits one to handle certain otherwise difficult topological problems associated with this group. If $f_1,f_2\in\mathcal{S}$ and $\vec{\varphi}_1,\vec{\varphi}_2$

$\in \mathrm{Diff}(R^3)$, the group law is

$$(f_1,\vec{\varphi}_1)(f_2,\vec{\varphi}_2) = (f_1+f_2 \circ \vec{\varphi}_1, \vec{\varphi}_2 \circ \vec{\varphi}_1). \tag{4}$$

Distinct unitary representations of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ describe distinct physical systems; for example, a system of N identical particles obeying Bose statistics and a system of N identical particles obeying Fermi statistics correspond to two particular unitarily inequivalent representations. Because $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ is an infinite-parameter group, it is possible for its unitary representations to describe a very wide class of physical situations. The gradual development of a representation theory suitable for such infinite-parameter groups has allowed many of these descriptions to be obtained.[3]

Next, let us introduce representations describing tightly bound composite quantum particles.[4] In the simplest case, the Hilbert space consists of square-integrable functions $\Psi(\vec{x},\vec{\lambda})$, where $\vec{x}$ is the position coordinate and $\vec{\lambda}$ the dipole moment coordinate. A unitary representation of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ is given by $U(f)V(\vec{\varphi})$, where

$$(U(f)\Psi)(\vec{x},\vec{\lambda}) = \exp[i\vec{\lambda}\cdot \mathrm{grad}\, f(\vec{x})]\Psi(\vec{x},\vec{\lambda}), \tag{5}$$

$$(V(\vec{\varphi})\Psi)(\vec{x},\vec{\lambda}) = \Psi(\vec{\varphi}(\vec{x}), \mathcal{J}_{\vec{\varphi}}(\vec{x})\vec{\lambda}) \det \mathcal{J}_{\vec{\varphi}}(\vec{x}), \tag{6}$$

and where $\mathcal{J}_{\vec{\varphi}}(\vec{x})_j{}^k = (\partial_j \varphi^k)(\vec{x})$ defines the Jacobian matrix of $\vec{\varphi}$ at $\vec{x}$. To interpret this representation, we can compare it with a two-particle representation of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ for ordinary particles of equal and opposite charge q:

$$(U(f)\Phi)(\vec{x}_1,\vec{x}_2) = \exp\{iq[f(\vec{x}_1) - f(\vec{x}_2)]\}\,\Phi(\vec{x}_1,\vec{x}_2), \tag{7}$$

$$(V(\vec{\varphi})\Phi)(\vec{x}_1,\vec{x}_2) = \Phi(\vec{\varphi}(\vec{x}_1), \vec{\varphi}(\vec{x}_2)). \tag{8}$$

When the particle separations are small, so that f and $\vec{\varphi}$ vary slowly between $\vec{x}_1$ and $\vec{x}_2$, Eqs. (7) and (8) approximate Eqs. (5) and (6) with $\vec{\lambda} = q(\vec{x}_1 - \vec{x}_2)$ and $\vec{x} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$. Thus $\Psi(\vec{x},\vec{\lambda})$ is a probability amplitude for finding a neutral particle at $\vec{x}$ with dipole moment $\vec{\lambda} \neq 0$. It is easy to generalize Eqs. (5) and (6) to representations of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ describing N identical dipole particles by taking symmetric or antisymmetric tensor products.

Similarly, a quadrupole particle may be represented in the Hilbert space of square-integrable functions $\Psi(\vec{x},\vec{\lambda},Q)$, where the quadrupole moment Q is a 3×3 symmetric tensor. For $\vec{\varphi} \in \mathrm{Diff}(R^3)$, define $\vec{\varphi}: (\vec{x},\vec{\lambda},Q) \to (\vec{x}',\vec{\lambda}',\vec{Q}')$ by

$$\vec{x}' = \vec{\varphi}(\vec{x}), \quad (\lambda')^k = (\partial_j \varphi^k)(\vec{x})\lambda^j + \tfrac{1}{2}Q^{mn}(\partial_m \partial_n \varphi^k)(\vec{x}),$$

$$(Q')^{mn} = (\partial_j \varphi^m)(\vec{x})(\partial_k \varphi^n)(\vec{x})Q^{jk},$$

summing over repeated indices. The configuration space can now be partitioned into nine nontrivial orbits under the action of $\mathrm{Diff}(R^3)$, labeled by the signs of the eigenvalues of Q. The quadrupole group representations are given by

$$(U(f)\Psi)(\vec{x},\vec{\lambda},Q) = \exp\left[i\lambda^j(\partial_j f)(\vec{x}) + \frac{i}{2}Q^{mn}(\partial_m\partial_n f)(\vec{x})\right]\Psi(\vec{x},\vec{\lambda},Q), \tag{9}$$

$$(V(\vec{\varphi})\Psi)(\vec{x},\vec{\lambda},Q) = \Psi(\vec{x}',\vec{\lambda}',Q')[(d\nu'/d\nu)(\vec{x},\vec{\lambda},Q)]^{1/2}, \tag{10}$$

where ν is a measure concentrated on one of the orbits whose measure-zero sets are preserved by the action of $\mathrm{Diff}(R^3)$, ν' is the measure transformed by $\vec{\varphi}$, and $d\nu'/d\nu$ is the Radon-Nikodym derivative. Each orbit leads to a different representation of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$. In analogy with the dipole case, quadrupole representations of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ approximate N-particle representations with $N = 3$ or more, for scalar functions and diffeomorphisms varying slowly between the particles' spatial coordinates. Thus, it seems physically plausible to regard the multipole representations given here as describing tightly bound composite nonrelativistic particles.

Now let us introduce a way to obtain particles with spin by means of representations of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$.[5] For fixed $\vec{x} \in R^3$, we define a map $h_{\vec{x}}: \mathrm{Diff}(R^3) \to \mathrm{SL}(3,R)$, where $\mathrm{SL}(3,R)$ is the group of real 3×3 matrices having determinant 1, given by $h_{\vec{x}}(\vec{\varphi}) = [\det \mathcal{J}_{\vec{\varphi}}(\vec{x})]^{-1/3}\mathcal{J}_{\vec{\varphi}}(\vec{x})$. Let Γ be a continuous path from infinity to $\vec{x}$ in R^3; then $h_\Gamma(\vec{\varphi})$ defines a continuous path from the identity to $h_{\vec{x}}(\vec{\varphi})$ in $\mathrm{SL}(3,R)$. Such a path defines an element of the universal covering group $\mathcal{SL}(3,R)$, so we have a map $\tilde{h}_{\vec{x}}: \mathrm{Diff}(R^3) \to \mathcal{SL}(3,R)$. This map can be used to *induce* representations of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ from representations of $\mathcal{SL}(3,R)$. Let π be a continuous unitary representation of $\mathcal{SL}(3,R)$ in a Hilbert space. The desired representation of $\mathcal{S} \wedge \mathrm{Diff}(R^3)$ acts in the Hilbert space of square-integrable $\mathfrak{M}$-valued functions on R^3 as follows:

$$(U(f)\Psi)(\vec{x}) = \exp[if(\vec{x})]\,\Psi(\vec{x}), \tag{11}$$

$$(V(\vec{\varphi})\Psi)(\vec{x}) = \pi(\tilde{h}_{\vec{x}}(\vec{\varphi}))\Psi(\vec{\varphi}(\vec{x}))[\det \mathcal{J}_{\vec{\varphi}}(\vec{x})]^{1/2}. \tag{12}$$

Now π can be decomposed with respect to SU(2), which is the maximal compact subgroup of $\mathcal{SL}(3, R)$. Following the notation of Sijacki,[6] let Σ_0 and $\Sigma_\pm$ represent the usual generators of SU(2) in $\mathfrak{M}$, and let T_μ, $\mu = -2, -1, 0, 1, 2$, represent the quadrupole generators of $\mathcal{SL}(3, R)$ in $\mathfrak{M}$, obeying $T_{-\mu}{}^* = (-1)^\mu T_\mu$. The representation π determines Σ_0, $\Sigma_\pm$, and T_μ. Define the functionals $G_\mu(\vec{g})$ as in Ref. 5.

The self-adjoint generators obtained from Eqs. (11)–(12) can now be written,

The self-adjoint generators obtained from Eqs. (11)–(12) can now be written,

$$(\rho(f)\Psi)(\vec{x}) = f(\vec{x})\Psi(\vec{x}), \tag{13}$$

$$(J(\vec{g})\Psi)(\vec{x}) = \frac{1}{2i}\{\vec{g}(\vec{x})\cdot\nabla\Psi(\vec{x}) + \nabla\cdot[\vec{g}(\vec{x})\Psi(\vec{x})]\} + \tfrac{1}{2}\,\mathrm{curl}\,\vec{g}(\vec{x})\cdot(\Sigma\Psi)(\vec{x}) + \tfrac{1}{2}\sum_{\mu=-2}^{2}[G_{-\mu}(\vec{g})](\vec{x})(T_\mu\Psi)(\vec{x}). \tag{14}$$

The first term in Eq. (14) describes orbital momentum density, the second describes spin momentum density, while the third is a spin-changing contribution which could occur for excited states of nuclei, for supermultiplets of hadrons lying on Regge trajectories, or in the presence of a strong, nonuniform gravitational field. Alternatively, one can construct from the foregoing operators a class of operators called "local rigid rotations," with respect to which the Hilbert space decomposes into invariant subspaces of fixed spin carrying representations of local SU(2); i.e., on such a subspace, there is a natural representation of the group $(\mathcal{S}\otimes\mathcal{T})\wedge \mathrm{Diff}(R^3)$, where $\mathcal{T}$ is the group of C^∞, SU(2)-valued functions on R^3 which tend rapidly toward the identity at infinity, and the group operation is pointwise multiplication. The self-adjoint generators of $\mathcal{T}$ are the spin density operators in the representation.

A continuous unitary representation of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$ is partially characterized by the set Δ of possible spatial configurations of the system it describes. The spatial configurations are to be thought of as generalized functions. For example, a configuration of N identical particles corresponds to $\sum_{j=1}^{N}\delta_{\vec{x}_j}$, where $\delta_{\vec{x}}$ denotes a δ function centered at $\vec{x}$. A dipole configuration associated with Eqs. (5) and (6) is of the form $-\vec{\lambda}\cdot\nabla_{\vec{x}}\delta_{\vec{x}}$, while a quadrupole configuration is of the form $(-\vec{\lambda}\cdot\nabla_{\vec{x}} + \frac{1}{2}Q^{mn}\partial_m\partial_n)\delta_{\vec{x}}$. The action of a diffeomorphism is to map one such configuration in Δ to another. Associated with a representation of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$ is a probability measure μ on Δ, allowing the outcome of an experiment to be expressed in terms of the probability that the configuration is in a specified set. The measure-zero sets of μ are preserved under $\mathrm{Diff}(R^3)$, and a "wave function" is given by a function on Δ square-integrable with respect to μ.

Representations of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$ have also been constructed for which Δ is infinite dimensional. For example, in the case of an infinite free Bose gas, a configuration may be written $\sum_{j=1}^{\infty}\delta(\vec{x}_j)$, where $\{x_j\}$ is characterized by its average density. Here the desired measure μ is a Poisson measure. It is a very attractive idea to obtain representations of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$ based on nonpointlike configurations. (See also Isham, Ref. 2.) For example, one can define a *string* configuration by means of a suitably smooth continuous path Γ in R^3, whose terminal points are $\vec{x}_1$ and $\vec{x}_2$, and a weight $w(\vec{x})$ defined on the path. The corresponding generalized function F is defined by

$$\langle F, f\rangle = \int_{\vec{x}_1\,\Gamma}^{\vec{x}_2} w(\vec{x})f(\vec{x})\,ds,$$

where ds is the path length. One can similarly define a *loop* configuration, in which $\vec{x}_1 = \vec{x}_2$. The space Δ of all string configurations (or alternatively all loop configurations) defines a formal representation of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$. It remains, however, to specify suitable measures on such spaces. Finally, we remark that representations based on configurations $\langle F, f\rangle = \int F(\vec{x})f(\vec{x})\,d^3x$, where the $F(\vec{x})$ are positive continuous functions, should be applicable to the quantization of a compressible fluid. The space Δ is here interpreted as the space of density functions.

For systems having infinitely many degrees of freedom, we expect different dynamics to be described by inequivalent representations of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$. Thus the dynamical problem of solving for the ground state associated with a given Hamiltonian becomes the problem of finding the correct representation of $\mathcal{S}\wedge\mathrm{Diff}(R^3)$.

The notion of a gauge group has a natural interpretation in the present setting. Given a spatial configuration $F\in\Delta$, consider the subgroup $\mathcal{K}_F$ of $\mathrm{Diff}(R^3)$ which leaves F invariant. Subject again to measure-theoretical considerations, unitary representations of $\mathcal{K}_F$ now *induce* representations of $\mathrm{Diff}(R^3)$ associated with the same set Δ of spatial configurations. If there is a natural homomorphism from $\mathcal{K}_F$ to a locally compact group $\mathcal{G}$, then unitary representations of $\mathcal{G}$ can be employed to induce representations of Diff(R^3), with fewer measure-theoretic problems. The group $\mathcal{G}$, or (more generally) $\mathcal{K}_F$ itself, serves as a gauge group for the theory, and its elements play the role of gauge transformations.

For the configurations describing N identical particles discussed above, $\mathcal{G}$ can be chosen to be the symmetric group S_N. Mathematically, this is the fundamental group of the space of N-particle configurations. The induced representations of S_N correspond to bosons or fermions. If R^2 replaces R^3 in the above development, $\mathcal{G}$ becomes the braid group B_N. Its one-dimensional unitary representations can describe not only bosons or fermions, but recently discussed particles obeying unusual statistics.[7] The case $\mathcal{G} = \mathcal{SL}(3, R)$ was discussed above. Similarly, gauge groups can be specified for string, loop, and fluid configurations.

The above results support the idea that Diff(R^3) can function as a kind of "universal group" for quantum theory, with its infinitesimal generators corresponding to noncanonical local quantum fields describing a wide variety of particles in inequivalent representations.

One of us (G.G.) thanks the Physics Department at Princeton University for support during his 1982–83 sabbatical year as a Visiting Fellow, and Los Alamos National Laboratory for its frequent hospitality. We are grateful to the U. S. Department of Energy for financial support.

[1]V. I. Arnold, Ann. Inst. Fourier (Grenoble) 16, 319 (1966); J. Marsden and A. Weinstein, "Coadjoint Orbits, Vortices, and Clebsch Variables for Incompressible Fluids" in Proceedings of the Los Alamos Conference "Order in Chaos," edited by D. K. Campbell, H. A. Rose, and A. C. Scott, Physica (Utrecht) (to be published; J. Marsden, A. Weinstein, T. Ratiu, R. Schmid, and R. G. Spencer, "Hamiltonian Systems with Symmetry, Coadjoint Orbits, and Plasma Physics," in Proceedings of the IUTAM Symposium on Modern Developments in Analytical Mechanics (to be published).

[2]J. A. Wheeler, "Geometrodynamics and the Issue of the Final State," in *Relativity, Groups, and Topology,* edited by B. S. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964); J. L. Friedman and R. D. Sorkin, Phys. Rev. Lett. 44, 1100 (1980), and Gen. Relativ. Gravit. 14, 615 (1982); C. J. Isham, "Quantum Geometry," to be published; J. E. Marsden, D. G. Ebin, and A. E. Fischer, "Diffeomorphism Groups, Hydrodynamics and Relativity," in *Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress,* edited by J. R. Vanstone (Canadian Mathematical Congress, Montreal, 1972).

[3]G. A. Goldin and D. H. Sharp, in *Group Representations in Mathematics and Physics,* edited by V. Bargmann, Battelle Seattle 1969 Rencontres (Springer, New York, 1970); G. A. Goldin, J. Math. Phys. (N.Y.) 12, 462 (1971); G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. (N.Y.) 21, 650 (1980).

[4]G. A. Goldin and R. Menikoff, to be published.

[5]G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Phys. A 16, 1827 (1983); G. A. Goldin and D. H. Sharp, Commun. Math. Phys. (to be published).

[6]D. Sijacki, J. Math. Phys. 16, 298 (1975). See also L. Weaver, L. C. Biedenharn, and R. Y. Cusson, Ann. Phys. (N.Y.) 77, 250 (1973).

[7]F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982); G. A. Goldin and D. H. Sharp, Phys. Rev. D 28, 830 (1983).