Baryon-Meson Mass Inequality

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It is suggested that the inequality $m_B > \frac{3}{2}m_M$ is a rigorous result in quantum chromodynamics. The analog for a $(q_1 \ldots q_N)$ baryon in SU(N) is $m_B > (\frac{1}{2}N)m_M$. The inequality is proved for weak coupling and a version of the strong-coupling expansion where a separation $H_{q_1 q_2 q_3} = H_{12} + H_{23} + H_{31}$ of the problem can be achieved. Implications for quantum chromodynamics and composite models are briefly discussed.

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(I) The baryon-meson mass ratios along with other approximate phenomenological $\frac{3}{2}$ factors served as early support for the quark model. I suggest that in QCD this relation transforms into a rigorous inequality:

$$m_B \geqslant \tfrac{3}{2}m_M, \tag{1}$$

connecting a ground-state baryon and the "corresponding" meson (e.g., φ for Ω^-) to be defined more precisely later.

While lattice techniques may soon facilitate reliable mass calculations,[1] the importance of exact relations can hardly be overemphasized. In addition, the inequality has far-reaching implications for composite models of quarks and leptons based on preons with gauge interactions.[2] The generalization of Eq. (1) to SU(N) is

$$m_B \geqslant (\tfrac{1}{2}N)m_M, \tag{2}$$

for baryons composed of N preons in the fundamental representation. This precludes in such models massless fermionic composites (due to unbroken flavor chiral symmetry) without massless bosons (which would naturally arise if this chiral symmetry was spontaneously broken!).

(II) To motivate the inequalities assume first that the dynamics of the three-quark system is describable by two-body interactions. Perturbatively this would be the one-gluon exchange. Nonperturbative infrared effects most likely modify the gluon's propagator.

I abstract only one aspect of the gluon exchange —the $\tilde{\lambda}_1 \cdot \tilde{\lambda}_2$ coupling. If this yields a potential $V_{q\bar{q}}$ between a quark and an antiquark when the latter is coupled to a color singlet, then

$$V_{qq} = [1/(N-1)]\,V_{q\bar{q}} \tag{3}$$

is the potential between any quark pair in the baryon. The $1/(N-1)$ overall ratio of color coefficients factors out completely and therefore the qq and $q\bar{q}$ should be in the same spin, orbital, etc., states.

Consider now the Hamiltonian for a $N=3$ baryon:

$$H_3(q_1q_2q_3) = T_1(q_1) + T_2(q_2) + T_3(q_3) + V_{q_1q_2} + V_{q_2q_3} + V_{q_3q_1}. \tag{4}$$

With use of $V_{qq} = \frac{1}{2}V_{q\bar{q}}$, $2H_3$ can be rearranged into a sum of three Hamiltonians for three $q_i\bar{q}_j$ subsystems:

$$2H_3 = (T_1 + T_2 + V_{12}{}^{q\bar{q}}) + (T_2 + T_3 + V_{23}{}^{q\bar{q}}) + (T_3 + T_1 + V_{31}{}^{q\bar{q}}) = H_{12} + H_{23} + H_{31}. \tag{5}$$

Let $\psi_3(1,2,3)$ be the lowest energy state of the three-quark system (ground-state baryon). Taking the matrix elements of (5) we have

$$2M_B = 2\langle \varphi_3 | H_3 | \psi_3\rangle = \langle \psi_3 | H_{12} | \psi_3\rangle + \langle \psi_3 | H_{23} | \psi_3\rangle + \langle \psi_3 | H_{31} | \psi_3\rangle. \tag{6}$$

ψ_3 is a wave function in the 1-2-3 space. However, since H_{12} (say) operates only on the variables of particles 1 and 2, $\langle \psi_3 | H_{12} | \psi_3\rangle$ is a sum of expectation values of the H_{12} between states in the 1-2 space only.

To show this let us formally expand

$$\psi_3 = \sum_{l,m,n} \varphi_l(1)\,\varphi_m(2)\,\varphi_n(3)\,C_{l,m,n}$$

with φ_l a complete one-particle set and $\langle \psi_3 | \psi_3\rangle = 1$ implying $\sum_{lmn} |C_{l,m,n}|^2 = 1$. Then as a result of

$$\langle \varphi_l(1)\,\varphi_m(2)\,\varphi_n(3) | H_{12} | \varphi_{l'}(1)\,\varphi_{m'}(2)\varphi_{n'}(3)\rangle = \delta_{nn'}\langle \varphi_l(1)\,\varphi_m(2) | H_{12} | \varphi_{l'}(1)\,\varphi_{m'}(2)\rangle, \tag{7a}$$

we have

$$\langle \psi_3 | \boldsymbol{H}_{12} | \psi_3 \rangle = \sum_n \langle \psi_2^n | H_{12} | \psi_2^n \rangle \tag{7b}$$

with

$$\psi_2^n = \sum_{l,m} C_{l,m,n} \varphi_l(1) \varphi_m(2).$$

Hence

$$\sum_n \langle \psi_2^n | \psi_2^n \rangle = 1. \tag{7c}$$

The variational principle implies for each of the expectation values in (7b)

$$\langle \psi_2^n | H_{12} | \psi_2^n \rangle \geq \boldsymbol{E}_{12} \langle \psi_2^n | \psi_2^n \rangle,$$

with $\boldsymbol{E}_{12}$ the lowest energy (or mass). Using (7b) and (7c)

$$\langle \psi_3 | \boldsymbol{H}_{12} | \psi_3 \rangle \geq \sum_n E_{12} \langle \psi_2^n | \psi_2^n \rangle \geq E_{12},$$

and finally we have the desired inequality

$$2M_B \geq M_{12} + M_{23} + M_{31}. \tag{8}$$

In Eq. (8), $M_{12}, M_{23}, \ldots$ are the masses of $q\bar{q}$ subsystems $(1,2),(2,3),\ldots$ in the angular momentum state, etc., imposed by the baryon. Thus, Δ^{++} consists of three u quarks all with parallel spins adding up to $S = \frac{3}{2}$. The corresponding meson states are $u\bar{u}$ triplet, i.e., the ρ or ω mesons. In both cases we have just s-wave quark-quark or quark-antiquark pairs. Equation (8) becomes[3]

$$m_{\Delta^{++}} \geq \tfrac{3}{2} m_\rho \ (1238 \geq 1155).$$

Likewise,

$$m_{\Omega^-} \geq \tfrac{3}{2} m_\varphi \ (1670 \geq 1520).$$

Also,

$$m_{ccc} \geq \tfrac{3}{2} m_\psi \ \ (m_{ccc} \geq 4.65 \text{ GeV}).$$

Likewise, for mixed flavors,

$$m_{\Xi^*} \geq \tfrac{1}{2}(m_\varphi + 2m_{K^*}) \ (1530 \geq 1400),$$

$$m_{\Sigma^*} \geq (\tfrac{1}{2} m_\rho + m_{K^*}) \ (1385 \geq 1280),$$

and for the charmed baryon analog,

$$m_{\Sigma^*{}_c} \geq \tfrac{1}{2}(2m_{D^*} + m_\rho) = 2.4 \text{ GeV}.$$

The SU(N) generalization is straightforward: With use of Eq. (3), $(N-1)H^{\text{baryon}}(q_1,\ldots,q_N)$ is reexpressed as a sum over all $\binom{N}{2}$ mesonic H_{ij} and the rest of the argument ensues.

In the nucleon we have two antiparallel spin pairs and hence Eq. (8) would imply $m_N \geq \frac{3}{4} m_\rho + \frac{3}{4} m_\pi$.[4]

The T_i and V_{ij} of Eq. (4) could be any relativistic or nonrelativistic spin-, coordinate-, energy-, etc., dependent one- and two-particle operators, respectively. Only the assumption of separability into two-body subsystems and the $(N-1)^{-1}$ Clebsch factor of Eq. (3) are required.

(III) The last separability assumption does not hold in the electric string confinement—"Y"—picture of baryons. I therefore rederive next the inequality in the strong-coupling limit.

I use the Hamiltonian lattice formulation,[5] though this section's argument could (and will) be given in the continuum. The Hamiltonian is

$$g^2 \sum_{\text{links}} \boldsymbol{E}^2 + \frac{1}{g^2} \sum_{\text{plaquettes}} \text{tr}(UUU^\dagger U^\dagger) + \text{H.c.} \tag{9}$$

Take first three heavy quarks located at $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

For strong coupling $g \to \infty$ the first term dominates and the energy is minimized by having the three quarks interconnected via a Y-like network of flux lines (three flux lines can join at a point via an ϵ_{abc} coupling), and $E_{3q} = \sigma \sum^3 |\vec{r}_i - \vec{x}|$ with $\vec{x}$ the junction point and σ the string tension.

The triangular inequality[6] implies $2E_{3q} \geq \sigma \times \sum_{i<j} |\vec{r}_i - \vec{r}_j|$. But the latter is just the total energy of three meson subsystems.

These energies should be introduced as potentials into the next step of an adiabatic slow-quark approximation. Having $2V_{3q} \geq V_{12} + V_{23} + V_{13}$ instead of $2V_3 = V_{12} + V_{23} + V_{13}$ as in the earlier section would reinforce the derived inequality (8).

The arguments generalize to SU(N) baryons; simple and repeated use of the triangular inequality shows that

$$(N-1)V_{q_1 \ldots q_N} = (N-1)\sigma \sum^n |\vec{r}_i - \vec{x}| \geq \sigma \sum_{i<j} |\vec{r}_i - \vec{r}_j| = \sum_{i<j} V_{ij}(q\bar{q}).$$

(IV) Having proved the inequality in the weak perturbative limit, and the other extreme of static quarks and strong coupling, I proceed to the case of static quarks but arbitrary couplings. Using states in the strong-coupling basis I describe the meson by a path functional $\sum_{\text{paths}} C(P_{12})|P_{12}\rangle$, i.e., a superposition of paths connecting $\vec{r}_1$ and $\vec{r}_2$ on the lattice.

The diagonal elements of the meson Hamiltonian are g^2 times the lattice length of the path. Off-diagonal elements of size $1/g^2$ connect paths which can be deformed into each other by the action of a single plaquette. Likewise, the baryon is given by the path functional $\sum_{\text{paths}} C(P_{123})|P_{123}\rangle$ and again all nonvanishing matrix elements connecting paths differing by a single plaquette equal $1/g^2$.

In addition, there are $1/g^2$ off-diagonal elements involving operations with plaquettes which raise the representation content of a given link, and/or generate new "junction points."

Even the approximation of keeping only minimally excited links and no string intersections or junctions except for the single junction required for the baryon represents a formidable problem. While neither meson or baryon energies can be exactly computed the comparison of the Hamiltonians can reinstate the inequality.

Within the approximation adopted here the Hamiltonian for the baryon $q_1q_2q_3$ problem[7] can be rewritten as the sum of the three $q_i\bar{q}_j$ Hamiltonians.[7] This is manifest for the diagonal terms by simply separating $2(P_{123})$ at the junction point, yielding $P_{1x2}+P_{2x3}+P_{3x1}$. Also any operation which does not directly involve the baryon junction modifies the path connecting 1 and x or 2 and x or 3 and x. In the first case, this operation is included in H_{13} and H_{12} as modification of a path connecting 1 and 3 and a path connecting 1 and 2. The same applies to the other two cases with appropriate cyclic permutations. This suggests that $H_{123}=H_{12}+H_{23}+H_{31}$. With use of the baryon ground-state functional $|B\rangle=\sum C(P_{123})\times|\boldsymbol{P}_{123}\rangle$ as a trial functional on the left- and right-hand sides of the last equation the procedure of deriving Eq. (8) from Eq. (5) can be repeated. In particular, since H_{12} does not operate on the string $(3,x)$ connecting to quark 3, we have in a self-explanatory notation

$$\langle P_{123}|\boldsymbol{H}_{12}|P_{123}'\rangle=\delta(P_{3x},P_{3x}')\langle P_{1x2}|H_{12}|P_{1x2}'\rangle.$$

The analog of Eq. (7) allows us to express the right-hand side as a sum of expectation values of $\boldsymbol{H}_{12}$, H_{23}, and H_{31} in just a $q\bar{q}$-type wave functional. With use of the normalization $\sum|C(P_{123})|^2=1$ and the variational argument for each expectation value the basic inequality (8) is rederived.

While any component (i.e., specific path P_{123}) of the baryon functional can be split as indicated above into a sum of three $q\bar{q}$ paths the reverse is not true. Only if the (12), (23), and (31) paths all intersect (at one single point in the present approximation) can they serve as a candidate for the baryon functional. This means that the correct operator relation is

$$2\boldsymbol{H}_{123}=P(H_{12}+H_{23}+H_{31})P^{\dagger} \qquad (10)$$

with P an appropriate projection operator.

This constraint on the trial states will enhance the inequality making it more difficult to minimize $\langle|H_{12}|\rangle$, $\langle|H_{23}|\rangle$, and $\langle|H_{31}|\rangle$ simultaneously. The triangular inequality of the last section is a simple manifestation of this effect in the strong-coupling limit.

(V) Instead of interpreting the energies for static quark potentials I simplify and generalize the derivation by introducing the kinetic energy terms $\sum\bar{\psi}_{\vec{n}}U_{\vec{n},\vec{n}+\hat{e}_i}\psi_{\vec{n}+\hat{e}_i}+\text{H.c.}$ This allows the quarks to move so that the $\vec{r}_i$ become dynamical variables rather than frozen parameters.

Neglecting pairs we can split this added piece into three parts $\sum^3 T_i$ according to which of the quarks 1, 2, or 3 is moved (T_1 includes also the mass term $m_1\psi_n{}^{\dagger}\psi_n$ for quark 1 etc.). From the analogy to Eq. (5) the addition of $2(T_1+T_2+T_3)$ to both sides of Eq. (10) will not change the inequality. In particular, since moving a quark at the end of one of three strings joining at a single junction x will still yield a configuration of the same type, the projection P is irrelevant for the T_i.

Thus, we have $2H_{123}'=P(H_{12}'+H_{23}'+H_{31}')P^{\dagger}$, with H' including also the quark kinetic terms, and again by following the procedure used above we obtain the inequality for this case as well.

The separation of $2H_B(1,2,3)$ utilized above is not evident for a baryon consisting of an interconnected maze of excited links.[8]

(VI) The variational arguments used in deriving the inequality apply to the sum of energies of the first n states. One may apply the inequalities to radial excitations relating N, roper resonance, and $\rho'\pi'$ states, etc. One could also try to relate $L\neq 0$ states in baryon and mesons.

Conceivably other rigorous inequalities can be derived for other quantities such as coupling constants, charge radii, or scattering lengths [it is amusing to note that another $\frac{3}{2}$ approximate relation $\sigma_{\text{tot}}(BB)\approx\frac{3}{2}\sigma_{\text{tot}}(M_B)$ is in effect phenomenologically an inequality: $\sigma_{\text{tot}}(BB)\gtrsim\frac{3}{2}\sigma_{\text{tot}}(M_B)$].

The inequality serves as another constraint on the program of obtaining the light quarks and leptons as bound states in a confining gauge theory which does not spontaneously break a chiral flavor symmetry.

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Note added.—After completing this work I learned from Herbert Neuberger that a weaker rigorous meson-baryon mass inequality was proved by Weingarten[9] using the Euclidean lattice approach. Subsequently, I became aware of related work by Vafa and Witten[10] and Witten[11] which does not, however, address the meson-baryon inequalities directly.

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[a]On leave of absence from Tel Aviv University, Ramat Aviv, Tel Aviv, Israel.

[1]For a recent review, see Michael Creutz, Laurence Jacobs, and Claudio Rebbi, Phys. Rep. 95, 201 (1983).

[2]G. E. 't Hooft, in *Recent Developments in Gauge Theories*, Proceedings of the Cargese Summer Institute, 1979, edited by G. E. 't Hooft *et al.* (Plenum, New York, 1980).

[3]Taking the ρ rather than the ω makes a little difference but avoids the gluon annihilation channel. Since this has no counterpart in the baryon the inequality strictly applies only with flavor nonsinglet mesons.

[4]The uu in the nucleon couples to $S=1$; the requirement of overall (uud) spin $\frac{1}{2}$ implies that each ud quark pair is in $S=1$ $(S=0)$ state with probability $\frac{1}{4}$ $(\frac{3}{4})$. This correction of a mistake in the original manuscript was kindly pointed out by S. L. Glashow.

[5]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).

[6]It holds for both the usual Euclidean and the "taxi-driver" metric of a lattice $r_{ij} \equiv |x_i - x_j| + |y_i - y_j| + |z_i - z_j|$.

[7]The lattice Hamiltonian is given by Eq. (9) with sums extending over all links and plaquettes. However, with a quark at $\vec{r}_1$ and an antiquark at $\vec{r}_2$ we can define the meson Hamiltonian as the links involved in the string network connecting 1 and 2 and the plaquette directly operating on such links. Disconnected vacuum loops should indeed be left out to obtain the proper meson energy. Alternatively, we can define the space of paths in which H_{12} operates by taking any simple string of minimally excited links joining 1 and 2 and operate on it any number of times with $\Sigma\, \mathrm{tr}(UUU^\dagger U^\dagger) + \mathrm{H.c.}$ without ever creating disjoint pieces. The baryon Hamiltonian is likewise defined in the space of strings interconnecting the three quarks.

[8]It appears that a weaker version, $m_B \geq m_M$, which is of little use for QCD but sufficient for the qualitative implications for composite models can be more readily obtained in this case. We just compare the energy of one $q\bar{q}$ pair, say at 1, 2, with that of the more complex and constrained baryon $q_1 q_2 q_3$ functional.

[9]D. Weingarten, Phys. Rev. Lett. 51, 1830 (1983).

[10]C. Vafa and E. Witten, Nucl. Phys. (to be published).

[11]E. Witten, to be published.