Numerical Evidence for $d_c=2$ in the Random-Field Ising Model

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A solid-on-solid interface representation of the random-field Ising model is studied numerically in two dimensions. The interface width varies linearly with sample size, in agreement with simple energy-accounting arguments and recent theories which predict that two is the lower critical dimension of the random-field Ising model.

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The critical and low-temperature (T) properties of several rather disparate systems, among them commensurate charge-density waves in the presence of random pinning centers[1] and dilute antiferromagnets in uniform magnetic fields,[2] can be well modeled by an Ising ferromagnet in a random magnetic field. This Letter reports numerical evidence aimed at distinguishing between competing theories of the low-T properties of such a random-field Ising model (RFIM). Our calculations support recent work by Grinstein and Ma,[3] and Villain,[4] who argue that the lower critical dimension, d_c, for this model is 2. Our results disagree with several recent papers[5,6] which concluded that $d_c=3$.

The controversy over d_c for this problem began with the "domain" argument of Imry and Ma,[7] who suggested that long-range ferromagnetic order is, even at $T=0$ and for arbitrarily weak random fields, unstable to the formation of large domains for $d<2$; i.e., $d_c=2$. Subsequently, however, discovery of a correspondence between the Ginzburg-Landau (GL) representation of the RFIM in d dimensions and the pure Ising model in $d-2$ dimensions suggested,[8] since the lower critical dimension for the pure system is 1, that $d_c=3$. Since the correspondence has been carefully established only for d near 6 this conclusion is, however, speculative.

More recent theoretical effort has focused on interface representations of the RFIM, i.e., models[3,4,6] devised for the study of a single domain wall separating domains of up and down Ising spins in the presence of random fields. The results of these studies have only intensified the debate. Pytte, Imry, and Mukamel[6] and Kogon and Wallace[6] have constructed and analyzed interface models based on the GL representation of the RFIM and concluded that $d_c=3$. The starting point for the interface models of Refs. 3 and 4 was, by contrast, discrete Ising spins. These authors argued $d_c=2$. Since all the interface calculations involve more or less plausible assumptions and uncontrolled approximations, the issue has not been definitively resolved. Experiments, interpretation of which has been complicated by irreversible effects[9] at low T, and computer simulations,[10] which are subject to severe finite-size limitations, have likewise failed to produce an unambiguous decision.

We describe here the first attempt to test directly the predictions of the various interface representations of the RFIM in two dimensions (2D). All of the interface models studied to date ignore "droplets" and "overhangs."[3,4,6] This common neglect makes these models comparatively easy to simulate; for a 2D Ising model, e.g., of size $L\times L$, there are only $(L-1)^L$ possible distinct positions of the interface separating up from down spins [i.e., in each column the interface can pass between any of the $(L-1)$ pairs of adjacent spins]. This is a tiny fraction of the $2^{L\times L}$ total states of the Ising model, and allows the numerical consideration of much larger L's ($L\sim 10^3$) than does straightforward Monte Carlo.

To define the lattice interface model studied here, consider an Ising model with nearest-neighbor interactions on a 2D square lattice. Let $i=0,\ldots,L$ label the columns and let the integer-valued function $f(i)$ define the position of an interface separating down spins from up. We impose boundary conditions $f(0)=f(L)=0$. The energy, H_0, associated with the broken bonds at the interface is $J[L+\sum_{i=0}^{L}|f(i+1)-f(i)|]$. An independent random field $h(i,j)$ sits on every site (i,j). Let

$$F[f(i)]\equiv \sum_{j=1}^{f(i)} h(i,j), \quad -\sum_{j=f(i)}^{-1} h(i,j), \quad 0 \qquad (1)$$

for $f(i)>0$, <0, $=0$, respectively. $H_1\equiv -2\times\sum_{i=1}^{L-1}F[f(i)]$ gives the random-field energy measured with respect to the energy of the flat interface, $f(i)=0$. The total Hamiltonian, H, is simply H_0+H_1. At each site, an independent random number, generated without bias between $-2h$ and $2h$, is assigned to $h(i,j)$. The average field strength

is then h.

The transfer-matrix method employed here starts from the definitions

$$V[f(i+1),f(i)] \equiv \exp\left(-\frac{J}{T}|f(i+1)-f(i)|+\frac{2F[f(i)]}{T}\right), \quad (2)$$

$$Y[f(n)] \equiv \sum_{f(n-1)} V[f(n),f(n-1)]\,Y[f(n-1)], \quad (3)$$

$$X[f(m)] \equiv \sum_{f(m+1)} V[f(m+1),f(m)]\,X[f(m+1)]; \quad (4)$$

$X[f(L)] \equiv Y[f(0)] = \delta_{f,0}$. $Y[f(n)]$ and $X[f(m)]$ can be evaluated recursively by starting at columns 1 and $(L-1)$, respectively. The quantity $Z[f(n)] \equiv Y[f(n)]X[f(n)]$ is proportional to the probability that the interface passes through the position $f(n)$. The partition function is given by $Z \equiv \sum_{f(n)} Z[f(n)]$, and the surface tension by $\sigma = -T\ln Z/L$. The thermally averaged mean square width at $L/2$, $w^2 \equiv \langle [f(L/2)]^2 \rangle$, is simply

$$\sum_{f(L/2)} [f(L/2)]^2 Z[f(L/2)]/Z.$$

The ratio of w^2 to $\bar{w}^2$, the *average* mean square width of *all* the columns, ought to approach a constant at large L. Numerically, we found $w/\bar{w}$ to be independent of L even for small L; we therefore present data only for w here. In carrying out the computations we chose the width, L', of the system (i.e., the maximum absolute value that f is allowed to take) to satisfy $L' > 10w$; we have found the results so obtained to be independent of L'.

Before displaying the results, let us estimate w for our model with crude energy-accounting arguments.[3,11] For an interface of length L and height w, $H_0 \sim J(L+2w)$ while the field energy $H_1 \sim -h(wL)^{1/2}$, for a total energy $E_D \sim J(L+2w) - h(wL)^{1/2}$. Minimizing E_D with respect to w yields the equilibrium value of w at $T=0$: $w_h \sim (h/4J)^2 L$. [Note that this result provides a rough lower bound on the sample lengths L required to achieve the "asymptotically large L" limit. The statistical considerations implicit in writing H_1 as $-h(wL)^{1/2}$ are obviously valid only for $w_h > 1$, or $L > (4J/h)^2 w_h$.] Since thermal fluctuations in 2D produce a width[12] $w_T \sim TL^{1/2}$, $w_h \gg w_T$ for large L, and so the effects of small finite T ought not to alter the $T=0$ results. As $w \sim L$ has been argued[3,4,6,11] to be the hallmark of the lower critical dimension, this energy accounting predicts $d_c = 2$ for the RFIM.

Let us compare the result $w \sim h^2 L$ with the 2D predictions of the *continuum* interface theories. References 3 and 4 argue $w \sim h^{2/3} L$. The linear L dependence of this result is identical to that of the energy-accounting prediction for the *lattice* model, as must be the case given that Refs. 3 and 4 also predict $d_c = 2$. The origin of the discrepancy between the h^2 and $h^{2/3}$ dependences is trivial; it derives from the difference between the expression, $J\int\{1+[\nabla f(x)]^2/2\}dx$, used to approximate the exchange energy across the interface in the model of Refs. 3 and 4, and the obvious continuum generalization of H_0, viz., $J\int[1+|\nabla f(x)|]dx$. [With $|\nabla f| \sim w/L$, the difference between $(\nabla f)^2$ and $|\nabla f|$ is readily identified, via the energy-accounting argument, as the source of the discrepancy.[13]] In the models of Ref. 6 (which assert $d_c = 3$), $w \sim hL^{3/2} g(h^2 L)$, but the behavior of the scaling function $g(z)$ at large z is unknown. Hypothesizing $g(z) \sim z^a$ as $z \to \infty$ for some exponent a, one obtains $w \sim h^{1+2a} L^{a+3/2}$, a prediction which can be tested by numerical study of *both* the h and L dependence of w. (Note that since the models of Ref. 6 have $w \sim L$ when $d = 3$, and since for a given L the interface ought to be *wider* in 2D than 3D, the exponent a must be $> -\frac{1}{2}$.)

Figure 1 shows a log-log plot of w vs L for $h = 0.125$ and a series of six temperatures between 1 and 0.05. Both h and T are measured in units of J. Two systematic trends are evident in this data. First, the data for the two highest temperatures show crossover from one slope at small L to a higher slope at larger L. The crossover is more gradual at $T=1$ than at $T=0.5$ and is easily understood as the crossover from widths which are essentially thermal in character [i.e., (Ref. 12), $w \sim TL^{1/2}$; we show as dashed lines in Fig. 1 the curves corresponding to $h=0$ for $T=1$ and 0.5] to widths dominated by the random fields (i.e., $w \sim L^x$, where all theories predict $x \geq 1$) at large L. Fitting only the data for $w \gtrsim 3$ for $T=0.5$ by the form $w \sim L^x$ we find that the best fit gives $x = 1.0$. (Numerical values of x quoted have errors of about 0.1.) Second, data at the lowest temperatures, for which the thermal widths are indiscernibly small, show the reverse trend—a slow crossover from a rather steep slope at *small* L to a lower one at higher L. We interpret this curious effect, which becomes more pronounced as T decreases, as follows: At low T, the thermal component of the width is negligible and, given the relatively weak random field $h = 0.125$, values of $L \sim 10$ are required to produce a nonzero w. As L increases beyond this "threshold," w shoots up rather more quickly than the linear dependence on L predicted for *large* L by the en-

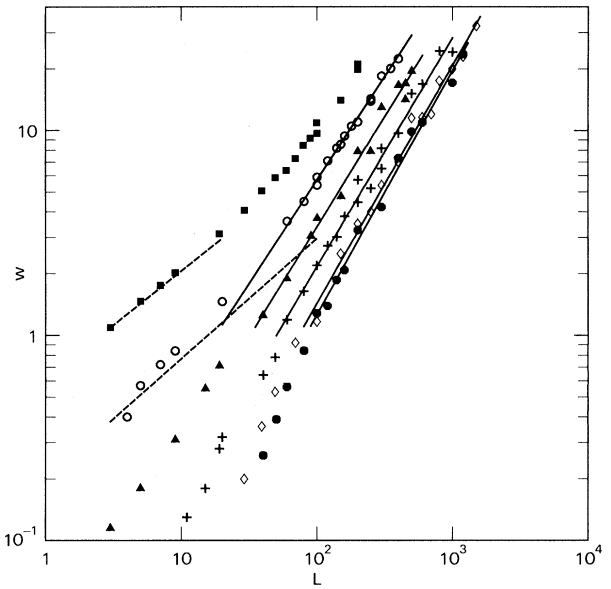

FIG. 1. Width w vs length L of the interface for $h = 0.125$ and various values of T: Solid circles for $T = 0.05$; diamonds for $T = 0.1$; plusses for $T = 0.2$; triangles for $T = 0.3$; open circles for $T = 0.5$; and squares for $T = 1$. The solid lines are best straight-line fits to the data for $w > 1$. The dashed lines correspond to $h = 0$. Both h and T are in units of J. All points shown correspond to averages over 100 realizations of $\{h\}$.

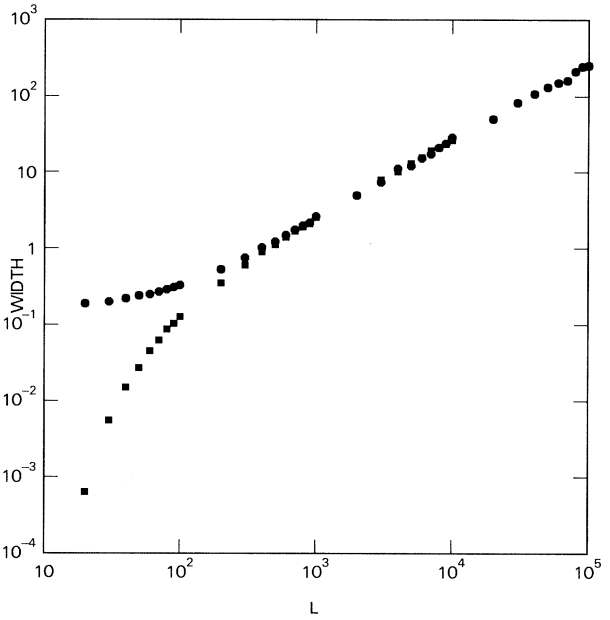

FIG. 2. Width vs L for the straight-interface model for $h = 0.125$; squares are for $T = 0.1$, and circles are for $T = 1$. All points shown correspond to 10^5 realizations of $\{h\}$.

ergy-accounting argument, and then slowly levels off[14] at larger L's. For each of the four lowest temperatures, the data for $w > 1$ appear to fall on a straight line. The best fit to these data give values of x ranging from 1.1 at $T = 0.3$ to 1.2 at $T = 0.05$. This systematic increase of x with decreasing T indicates that, the apparent linearity of the curves for $w > 1$ notwithstanding, at least some of the data for $w > 1$ are still in the regime of crossover from the steep initial rise. As evidence for this hypothesis we also fitted only the data for $w \geq 5$ at each of the four lowest temperatures, obtaining x's ranging between 1.0 at $T = 0.3$ and 1.1 at $T = 0.05$; these numbers are indeed significantly lower than those obtained from the $w > 1$ range of data. Acquiring data for sample sizes larger than our maximum L of 10^3 is computationally demanding. We have, however, tested our interpretation of the data on a somewhat simpler model, one whose Hamiltonian H is identical to the one we have been considering, but in which the interface is constrained to be rigidly horizontal: $f(i) = f(1)$ for all $1 \leq i \leq (L-1)$. The exchange energy H_0 for a given interface width w $[= f(i)]$ is then rigorously given by $J(L+2w)$, whereupon $w \sim h^2 L$ must hold exactly at large L.

Figure 2 shows a log-log plot of w vs L data for $h = 0.125$ at $T = 1$ and 0.1 for this model, whose simplicity allows treatment of extremely large ($L \sim 10^5$) systems. The upper ($T = 1$) curve shows the same crossover observed in Fig. 1 from a thermally dominated regime at small L (in this simplified model the thermal width is proportional to T and independent of L) to the random-field behavior, $w \sim L$, at large L. The lower curve ($T = 0.1$, where the thermal width is negligible) shows the threshold effect, i.e., the rather steep initial rise, that we saw in the more realistic model; the slope then decreases. The best fit to the data for L between 10^3 and 10^5 is $x = 1.0$. However, the best fit to the data for L between 10^2 and 10^3 gives $x = 1.27$, a considerably higher value. On this basis, we feel confident in asserting that the apparent increase of x (from its $T = 1$ value of unity) with decreasing T in Fig. 1 is an artifact of the threshold effect.

Representative data to test the dependence of w on h are displayed in Fig. 3, a log-log plot of w vs h for $T = 0.1$ and $L = 200$. The data points fall on a straight line with slope $y = 2.0 \pm 0.1$, consistent with the energy-accounting arguments which predict $w \sim h^2 L$. This value of y is difficult to reconcile with the theories of Ref. 6 which predict $d_c = 3$. Suppose that we attribute an exponent

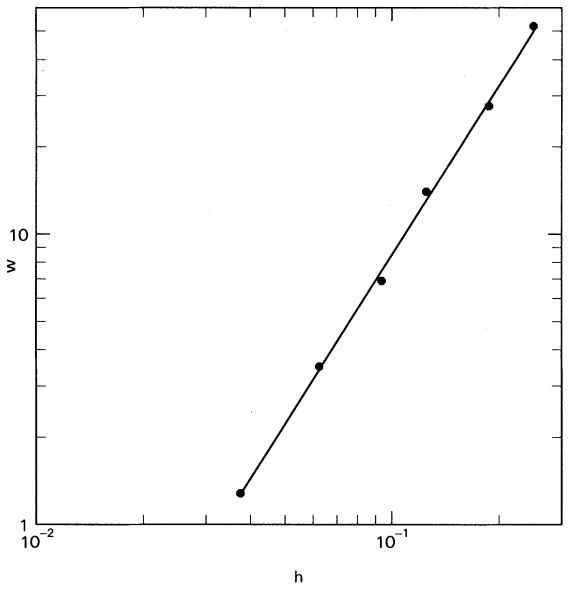

FIG. 3. W vs h for $L=200$ and $T=0.1$. All points shown correspond to 100 realizations of $\{h\}$.

$x=1.2$, the biggest exponent that our data can possibly allow, to the w vs L curves. This choice corresponds to $a=-0.3$ in the formula $w \sim h^{1+2a} \times L^{a+3/2}$, whereupon $w \sim h^{0.4}$, a dependence which our w vs h data will simply not admit.

In summary, our numerical calculations for a 2D lattice interface model with random magnetic fields support simple energy-accounting arguments which predict $w \sim L$ in 2D and hence that $d_c = 2$ in the RFIM. It would be interesting to test these predictions further by extending the computations described here to 3D.

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[12]See, e.g., D. J. Wallace, in *Phase Transitions, Cargese 1980,* edited by M. Levy, J. C. LeGuillou, and J. Zinn-Justin (Plenum, New York, 1982), p. 423.

[13]This discrepancy has a clear analog in the 2D *pure* Ising interface problem. There it is easy to show that exchange energies of the form $J\Sigma|f(i+1)-f(i)|$ and $J\Sigma|f(i+1)-f(i)|^2$, respectively, give rise to $w \sim TL^{1/2}$ and $w \sim (TL)^{1/2}$. Thus, the T dependence changes while the L dependence does not. See, e.g., Ref. 12.

[14]The case of $T = 0.3$ does not manifest either the thermal or the threshold effect clearly, though a weak thermal tail is visible at small L.