PHYSICAL REVIEW LETTERS

VOLUME 51 | 28 NOVEMBER 1983 | NUMBER 22

Lattice Fermions and Tomography

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(Received 31 August 1983)

The tomographic transform is combined with Susskind-like methods for lattice fermions to yield a new formulation with no multiplicity of species and with the usual discrete chiral symmetry.

PACS numbers: 11.15.Ha, 11.30.Rd

It is well known that the naive way of putting fermions on the lattice produces a multiplicity of species—sixteen, to be precise, in four dimensions.[1] Wilson's way[1] of handling the problem was to add extra terms to the action so as to give the fifteen unwanted fermions masses of the order of the inverse lattice spacing. In this way they become infinite in the continuum limit. This method is in some sense uneconomical—there are several redundant degrees of freedom. Moreover, there is no chiral symmetry. The Susskind method[2,3] and its reduced version[3,4] use fewer variables per lattice site and keep a discrete chiral symmetry, but some multiplicity still remains. In the unreduced case, there are four fermions ("flavors") and there is a continuous U(1) symmetry corresponding to a special chiral-flavor rotation. A Weyl reduction with respect to this produces reduced Susskind fermions—the multiplicity now is two.

I show how a complete reduction can be made and a lattice action describing a *single* fermion written down. The technique used for this purpose is Sommerfield's tomographic transform.[5] This is a way of representing four-dimensional fields (on the continuum) as two-dimensional fields carrying an index which runs over a continuous infinity of values. Instead of applying the reduced Susskind discretization on the usual four-dimensional continuum action, we first carry out this nonlocal transformation and put the two-dimensional fermions on the lattice. The point is that in *two* dimensions, reduced Susskind fermions come with a multiplicity of one.

Sommerfield[5] defines the tomographic transform of the fermion field by

$$\tilde{\psi}^{\alpha}(y,\hat{n}) = (2\pi)^{-1}\int d^3r\, \delta'(y_1 - \hat{n}\cdot\vec{r})u^{\alpha\dagger}(\hat{n})\psi(y_0,\vec{r}). \quad (1)$$

Here $y=(y_0,y_1)$, its two components being allowed to take arbitrary real values. y_0 denotes the time, here understood to be Euclidean; y_1 is the spatial coordinate of the two-dimensional fermion. $\hat{n}$ is a unit vector in three-dimensional space, in which $\vec{r}$ is the radius vector for the four-dimensional fermion. Clearly, Lorentz invariance is not manifest. The index α can take the two values 1,2. $u^{\alpha}(\hat{n})$ is a normalized four-component c-number spinor satisfying the eigenvalue conditions

$$\vec{\alpha}\cdot\hat{n}\,u^{\alpha}(\hat{n}) = u^{\alpha}(\hat{n}), \quad \vec{\Sigma}\cdot\hat{n}\,u^{\alpha}(\hat{n}) = -(-)^{\alpha}u^{\alpha}(\hat{n}), \quad (2)$$

where the vector $\vec{\alpha}$ consists of the three Hermitian matrices

$$\alpha_i = i\gamma_i\gamma_0, \quad (3)$$

and

$$\vec{\Sigma} = -(i/2)\vec{\alpha}\times\vec{\alpha}. \quad (4)$$

We make a slight modification of the tomographic transform and introduce

$$\hat{\psi}^1(y,\hat{n}) = \tilde{\psi}^1(y,\hat{n}), \quad \hat{\psi}^2(y,\hat{n}) = \tilde{\psi}^2(\mathcal{P}y,\mathcal{P}\hat{n}), \quad (5)$$

where $\mathcal{P}$ changes the sign of y_1 and $\hat{n}$. One can then show that the equations of motion of the (massless) Dirac field ψ can be rewritten in terms of these fields as

$$(\partial_0 + i\sigma_3\partial_1)\hat{\psi}(y,\hat{n}) = 0. \tag{6}$$

The Euclidean action may be taken to be

$$S = \int d^2\hat{n} \int d^2y \bar{\hat{\psi}}(y,\hat{n})(-\sigma_2\partial_0 + \sigma_1\partial_1)\hat{\psi}(y,\hat{n}), \tag{7}$$

which exhibits the two-dimensional Dirac structure clearly. The mass terms spoil the simplicity of (7) and will not be considered here; see Ref. 5 for details. The important point about the tomographic transform is that only y_0 and y_1 derivatives enter the action, so that $\hat{n}$ can be treated as an internal symmetry index, and one has essentially a two-dimensional space-time. Note also that the transformation is linear, so that the Jacobian introduced into the functional integral is a constant.

The naive way of transcribing the action (7) on a rectangular lattice would involve replacing the derivatives ∂_μ by central differences ∇_μ, defined by

$$\nabla_\mu f(y) = \tfrac{1}{2}\{f(y+e_{\hat{\mu}}) - f(y - e_{\hat{\mu}})\}, \tag{8}$$

where e_μ denotes a shift of the μ coordinate by one unit, the lattice spacing being taken to be unity. The action obtained in this way would really describe four fermions (in two dimensions), as is well known. We avoid this multiplicity altogether by use of reduced Susskind fermions.[3,4] In the conventional formulation of these fermions, one has a single one-component variable associated with every site. But like ordinary Susskind fermions, they too can be described in terms of a block lattice[6] of spacing double the usual one, and then one has (in two dimensions) a spinor and an antispinor at each lattice site. The action differs from the naive one by certain "irrelevant" pieces which vanish in the naive continuum limit but are effective on the lattice in avoiding multiplication of species. The appropriate expression in the present case reads

$$S_{\text{lattice}} = \int d^2\hat{n} \textstyle\sum_y \bar{\hat{\psi}}(y,\hat{n})(-\sigma_2\nabla_0 + \sigma_1\nabla_1 - \Delta_0 - i\sigma_3\Delta_1)\hat{\psi}(y,\hat{n}), \tag{9}$$

where the operators Δ_μ are defined by

$$\Delta_\mu f(y) = \tfrac{1}{2}[f(y+e_\mu) + f(y - e_\mu) - 2f(y)]. \tag{10}$$

The momentum-space poles of the propagator given by (9) correspond to

$$\sin^2(k_0/2) + \sin^2(k_1/2) = 0, \tag{11}$$

which is known to be free from doubling.

For the introduction of gauge interactions, one has to go to the alternative description. It is of course possible to replace the "derivatives" in (9) by gauge-covariant derivatives, but this manner of introducing an interaction violates the discrete chiral symmetry of the action and may lead to mass counterterms.[7] The other description uses the lattice which has sites at both integral and half-integral values of y_0 and y_1. One introduces the new one-component variable χ through the relations

$$\hat{\psi}(y,\hat{n}) = \begin{pmatrix} i\chi(y+\frac{1}{2}e_0,\hat{n}) + \chi(y+\frac{1}{2}e_1,\hat{n}) \\ \chi(y+\frac{1}{2}e_0,\hat{n}) + i\chi(y+\frac{1}{2}e_1,\hat{n}) \end{pmatrix},$$
$$\bar{\hat{\psi}}(y,\hat{n}) = (-i\chi(y,\hat{n}) - \chi(y+\tfrac{1}{2}e_0+\tfrac{1}{2}e_1,\hat{n}) \quad \chi(y,\hat{n}) + i\chi(y+\tfrac{1}{2}e_0+\tfrac{1}{2}e_1,\hat{n})). \tag{12}$$

Then (9) can be rewritten as

$$S_{\text{lattice}} = 2\int d^2\hat{n} \textstyle\sum_y \sum_\mu \eta_\mu(y)\chi(y,\hat{n})\chi(y+\frac{1}{2}e_\mu,\hat{n}), \tag{13}$$

where the sum over y now runs over integral and half-integral values, and

$$\eta_0(y) = 1, \quad \eta_1(y) = (-)^{2y_0}. \tag{14}$$

Gauge invariance needs $\chi(y+\frac{1}{2}e_\mu,\hat{n})$ in (13) to be parallel transported to y. From (12), one sees that for *complex* fermions, sites with one-half-integral coordinate are to be treated differently from sites with an even number of such coordinates. Introducing the function

$$\epsilon(y) = (-)^{2(y_0+y_1)} \tag{15}$$

we can write down the gauge-invariant action[4] as

$$S_{\text{lattice}}^{\text{g.i.}} = 2\int d^2\hat{n} \sum_y \sum_\mu \eta_\mu(y)\chi^T(y,\hat{n})\left[\frac{1+\epsilon(y)}{2}U(y,\mu,\hat{n}) + \frac{1-\epsilon(y)}{2}U^*(y,\mu,\hat{n})\right]\chi(y+\tfrac{1}{2}e_\mu,\hat{n}). \tag{16}$$

Here the U's belong to some gauge group of matrices and act on the χ's. Local transformations

$$\chi(y,\hat{n}) \to \left[\frac{1+\epsilon(y)}{2}S^*(y,\hat{n}) + \frac{1-\epsilon(y)}{2}S(y,\hat{n})\right]\chi(y,\hat{n}), \quad U(y,\mu,\hat{n}) \to S(y,\hat{n})U(y,\mu,\hat{n})S^{-1}(y+\tfrac{1}{2}e_\mu,\hat{n}) \tag{17}$$

leave this action invariant. This is true for both Abelian and non-Abelian groups, but it is only in the Abelian case that the $(y,\hat{n})$-dependent transformations are equivalent to local gauge transformations in the $(y_0,\vec{r})$ space in the continuum limit. The usual non-Abelian gauge transformations, local in the $(y_0,\vec{r})$ space, correspond to nonlocal transformations in tomographic coordinates and will not be considered here. We restrict the present discussion of the gauge-field part of the action also to the Abelian case.

The tomographic transform of the vector field $(A_0,\vec{A})$ consists of the following components[5]:

$$\tilde{A}_0(y,\hat{n}) = (2\pi)^{-1}\int d^3r\, \delta'(y_1 - \hat{n}\cdot\vec{r})A_0(y_0,\vec{r}), \quad \tilde{A}_1(y,\hat{n}) = (2\pi)^{-1}\int d^3r\, \delta'(y_1-\hat{n}\cdot\vec{r})\hat{n}\cdot\vec{A}(y_0,\vec{r}),$$
$$\tilde{A}_\alpha^T(y,\hat{n}) = (2\pi)^{-1}\int d^3r\, \delta'(y_1-\hat{n}\cdot\vec{r})\hat{\epsilon}_\alpha(\hat{n})\cdot\vec{A}(y_0,\vec{r}). \tag{18}$$

Here α runs over 1,2 and for each $\hat{n}$ there are two unit vectors $\hat{\epsilon}_\alpha(\hat{n})$ forming a right-handed triad with $\hat{n}$ and satisfying $\hat{\epsilon}_\alpha(-\hat{n}) = -(-)^\alpha\hat{\epsilon}_\alpha(\hat{n})$. The massless vector field action can be expressed as

$$S^{\text{gauge}} = -\tfrac{1}{4}\int d^2\hat{n}\int d^2y\,[\partial_\mu\tilde{A}_\alpha^T(y,\hat{n})\partial_\mu\tilde{A}_\alpha^T(y,\hat{n}) + \{\partial_0\tilde{A}_1(y,\hat{n}) - \partial_1\tilde{A}_0(y,\hat{n})\}^2]. \tag{19}$$

A gauge transformation

$$A_0(y_0,\vec{r}) \to A_0(y_0,\vec{r}) + \partial_0\theta(y_0,\vec{r}), \quad \vec{A}(y_0,\vec{r}) \to \vec{A}(y_0,\vec{r}) + \nabla\theta(y_0,\vec{r}) \tag{20}$$

leaves $\tilde{A}_\alpha^T$ unaffected, while

$$\tilde{A}_\mu(y,\hat{n}) \to \tilde{A}_\mu(y,\hat{n}) + \partial_\mu\tilde{\theta}(y,\hat{n}), \tag{21}$$

where

$$\tilde{\theta}(y,\hat{n}) = (2\pi)^{-1}\int d^3r\, \delta'(y_1-\hat{n}\cdot\vec{r})\theta(y_0,\vec{r}). \tag{22}$$

The part of the action involving $\tilde{A}_\alpha^T$ can be discretized in the same way as the action for a scalar field. The other piece is the gauge-field action for a two-dimensional theory and can be treated in the usual way. Instead of the $\tilde{A}_\mu$, one uses the phase factors $U(y,\mu,\hat{n})$ associated with the links of the two-dimensional lattice and writes down the interaction in, for example, the Wilson form by taking products of the U's around plaquettes:

$$S_{\text{lattice}}^{\text{gauge}} = -\tfrac{1}{16}\int d^2\hat{n}\sum_y \delta_\mu^+\tilde{A}_\alpha^T(y,\hat{n})\delta_\mu^+\tilde{A}_\alpha^T(y,\hat{n}) + (1/4e^2)\int d^2\hat{n}\sum_{\text{plaquettes}}\text{Re}\{U_p(\hat{n}) - 1\}. \tag{23}$$

Here the derivatives $\delta_\mu^\pm$ are defined by

$$\delta_\mu^\pm f(y) = \pm 2\{f(y\pm\tfrac{1}{2}e_\mu) - f(y)\} \tag{24}$$

and some unexpected powers of 2 appear because the lattice spacing is $\frac{1}{2}$.

The action consisting of the pieces (16) and (23) is invariant under

$$\chi(y,\hat{n}) \to (-)^{2y_1}\chi(y+\tfrac{1}{2}e_0+\tfrac{1}{2}e_1,\hat{n}),$$
$$U(y,\mu,\hat{n}) \to U(y+\tfrac{1}{2}e_0+\tfrac{1}{2}e_1,\mu,\hat{n}) \tag{25}$$

($\tilde{A}_\alpha^T$ unchanged). This discrete transformation reduces to a $\pi/2$ chiral rotation in the continuum limit, so that mass counterterms are ruled out.

To summarize, we have combined the tomographic transform and the reduced Susskind method to formulate a lattice action involving no multiplication of species and possessing a discrete chiral symmetry. Its advantage over the Wilson action lies in its chiral symmetry and the economical use of fermion variables. Unlike the usual Susskind action, it describes a single fermion. There is some nonlocality entering through the tomographic transformation, but it is an innocent one, unlike that in the method of Drell, Weinstein, and Yankielowicz,[8] because here it occurs already in the continuum and not in the passage to the lattice. The action proposed by Jacobs[9] shares all these advantages and has, in addition, a continuous chiral symmetry as against the merely discrete one that we have. But there is a price to be paid, namely the introduction of a quenched, random field that decouples in the continuum limit. The only disadvantages of the present formulation seem to be those inherent in the tomographic transform: lack of manifest Lorentz covariance (in the naive continuum limit) and the complicated, nonlocal look of non-Abelian interactions, which have in fact been omitted for simplicity. We cannot rule out the possibility that these features mess up the *quantum* continuum limit, but hope that the usual ideas of universality hold.

The author would like to thank Professor Abdus Salam for hospitality at the International Centre for Theoretical Physics, Trieste.

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