Energetic Particle Stabilization of Ballooning Modes in Tokamaks

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Introduction of an anisotropic, highly energetic trapped-particle species into a tokamak may allow direct stable access to the high-beta regime of second stability. Under certain conditions, the mode at marginal stability acquires a real frequency close to the precessional drift frequency of the energetic particles, perhaps correlating with recent "fishbone" observations on PDX.

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Plasma stabilization by an energetic particle component has been proposed and analyzed in the Astron[1] and ion ring devices[2,3] and in the ELMO bumpy torus.[4] In the latter, annuli of hot electrons provide stability for the toroidal core plasma. Because these hot electrons precess so rapidly, they tend to be rigid with respect to usual $E\times B$ fluid displacements and hence create a stabilizing diamagnetic well. In this Letter we suggest that energetic particles could have similar value if introduced into a tokamak. Whereas continuous introduction of hot particles is essential for stability in the bumpy torus, in a tokamak they may only be required until the plasma reaches the second stability region[5,6] where stability may improve with increasing beta, as has been shown at least with respect to ballooning and internal kink modes.

The stabilizing effects of fast ions have been pointed out recently by Connor *et al.*,[7] who analyzed isotropic circulating particles in the zero-bounce-frequency limit. The ballooning stability of anisotropic tokamaks has also been examined, without kinetic effects[8] but with finite gyroradii.[9]

Here, we analyze ideal magnetohydrodynamic (MHD) ballooning stability when a fairly anisotropic population of energetic particles is mirror trapped on the unfavorable-curvature side of a tokamak. These particles are assumed to drift across field lines rapidly: $\overline{\omega}_{dh} \gg |\omega|$, where $\overline{\omega}_{dh}$ is their bounce-averaged magnetic drift frequency and ω is the frequency (or growth rate) for the perturbation of interest. Also, since they are trapped on the outside, we assume that $\omega_{*h}/\overline{\omega}_{dh} > 0$, with ω_{*h} their diamagnetic frequency.

Under these assumptions, we can investigate linear stability by means of the low-frequency kinetic energy principle[10,11] $\delta W = \delta W_f + \delta W_k$, where the fluid term is

$$\delta W_f = \tfrac{1}{2}\int (ds/B)\{\sigma|\nabla S|^2(\hat{b}\cdot\nabla\Phi)^2 + \tau[Q_\parallel - (\sigma/\tau)B\vec{e}\cdot\vec{\kappa}\Phi]^2 - (\vec{e}\cdot\vec{\kappa})[\vec{e}\cdot\tilde{\nabla}P_\parallel + (\sigma/\tau)\vec{e}\cdot\tilde{\nabla}P_\perp]\Phi^2\} \tag{1}$$

and the kinetic term (for the non-MHD energetic species) is

$$\delta W_k = \tfrac{1}{2}\int dE\, d\mu\, \vec{e}\cdot\nabla F_h \frac{[\int(ds/v_\parallel)(\mu Q_\parallel + v_\parallel{}^2\vec{e}\cdot\vec{\kappa}\Phi)]^2}{\int(ds/v_\parallel)(\mu\vec{e}\cdot\nabla B + v_\parallel{}^2\vec{e}\cdot\vec{\kappa})}. \tag{2}$$

Here, $Q_\parallel$ is the (Lagrangian) magnetic field perturbation parallel to the equilibrium field $\vec{B} = \hat{b}B$, and Φ is the perturbed electrostatic potential; $P_{\perp,\parallel}$ are the total pressure components; s is the arc length along a field line, and $\tilde{\nabla} = \nabla - (\nabla B)\partial/\partial B$, $\sigma = 1 + (P_\perp - P_\parallel)/B^2$, $\tau = 1 + (\partial P_\perp/B\partial B)$, $\vec{\kappa} = (\hat{b}\cdot\nabla)\hat{b}$, $\mu = v_\perp{}^2/2B$, and $E = v_\parallel{}^2/2 + \mu B$. We have restricted attention to high-mode-number interchange-ballooning modes, whose transverse variation is in the eikonal S, where $\hat{b}\cdot\nabla S = 0$ and $\vec{e} = \vec{B}\times\nabla S/B^2$. Equation (2) pertains to the high-bounce-frequency limit, appropriate for trapped fast particles, in which their distribution function F_h is constant on a field line. Hot particles trapped on the outside of a tokamak stabilize through δW_k, but are destabilizing in δW_f.

To simplify the analysis of δW_k, we invoke the Schwartz inequality to obtain a lower bound: $\delta W_k \geqslant \delta W_1$, with

$$\delta W_1{}^{(j)} = \frac{1}{2}\frac{\{\int(ds/B)[(Q_\parallel/B)\vec{e}\cdot\tilde{\nabla}P_{\perp h} + (\vec{e}\cdot\vec{\kappa})(\vec{e}\cdot\tilde{\nabla}P_{\parallel h})\Phi]\}^2}{\int(ds/B)[B^{-1}(\vec{e}\cdot\nabla B)(\vec{e}\cdot\tilde{\nabla}P_{\perp h}) + (\vec{e}\cdot\vec{\kappa})(\vec{e}\cdot\tilde{\nabla}P_{\parallel h})]}. \tag{3}$$

A pessimistic estimate of stability can then be obtained by first minimizing $\delta W_f + \delta W_1$ with respect to

$Q_\parallel$ to obtain $Q_\parallel = (\sigma/\tau)B\Phi(\vec{e}\cdot\vec{\kappa}) - (1/\tau B)(\vec{e}\cdot\tilde{\nabla}P_{\perp h})\Lambda$, with

$$\Lambda^{(j)} = \frac{\int (ds/B)(\vec{e}\cdot\vec{\kappa})[\vec{e}\cdot\tilde{\nabla}P_{\parallel h} + (\sigma/\tau)\vec{e}\cdot\tilde{\nabla}P_{\perp h}]\Phi}{\int (ds/B)\{(\vec{e}\cdot\vec{\kappa})[\vec{e}\cdot\tilde{\nabla}P_{\parallel h} + (\sigma/\tau)\vec{e}\cdot\tilde{\nabla}P_{\perp h}] - (1/\tau B^2)(\vec{e}\cdot\tilde{\nabla}P_{\perp h})(\vec{e}\cdot\nabla P_c)\}}, \tag{4}$$

where $P_{\perp h}$ and P_c are the hot and (isotropic) core plasma pressures. The line integrals in Eqs. (3) and (4) are to be performed over the jth trapped-particle region. Next we vary with respect to Φ to obtain the integrodifferential ballooning equation

$$\vec{B}\cdot\nabla[(\sigma|\nabla S|^2/B^2)\vec{B}\cdot\nabla\Phi] + (\vec{e}\cdot\vec{\kappa})[\vec{e}\cdot\tilde{\nabla}P_\parallel + (\sigma/\tau)\vec{e}\cdot\tilde{\nabla}P_\perp]\Phi = (\vec{e}\cdot\vec{\kappa})[\vec{e}\cdot\tilde{\nabla}P_{\parallel h} + (\sigma/\tau)\vec{e}\cdot\tilde{\nabla}P_{\perp h}]\Lambda. \tag{5}$$

The general solution of Eq. (5) is $\Phi = \Phi_0 + c\Phi_1$, where Φ_0 is the homogeneous solution and Φ_1 is the particular solution for $\Lambda = 1$, with c then determined by Eq. (4). The solution is successively generated by solving for Φ_0 and Φ_1 in each trapped/untrapped region. If Φ is well behaved at infinity and does not change sign for $|s| < \infty$, the equilibrium is stable.

Note that for small core-plasma beta, the right-hand side of Eq. (5) can be expanded to show that ballooning instability is then driven only by the core pressure gradient, while the hot particles contribute a stabilizing diamagnetic well. For large β_c, the integrand in the denominator of Λ can vanish; this is related to the core beta limit predicted for bumpy tori[10] but for our problem occurs after drift reversal.

In order to apply Eq. (5) to our stability problem, we must first obtain an appropriate equilibrium. With $\vec{B} = \nabla\psi\times\nabla\beta$ in Clebsch form, the poloidal flux ψ satisfies the anisotropic Grad-Shafranov equation[12]

$$\left(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right)\psi + \nabla\psi\cdot\nabla\ln\sigma = -\frac{1}{\sigma^2}\frac{\partial G}{\partial\psi} - \frac{R^2}{\sigma}\frac{\partial P_\parallel}{\partial\psi}, \tag{6}$$

with $G(\psi) = \frac{1}{2}(\sigma R B_T)^2$, $B_T = R\vec{B}\cdot\nabla\varphi$ the toroidal field with φ the toroidal angle of symmetry, and R and Z the major radius and symmetry axis coordinates. Parallel pressure balance requires $\hat{b}\cdot\nabla(P_\parallel/B) = -(P_\perp/B^2)\hat{b}\cdot\nabla B$. We consider large-toroidal-mode-number ballooning modes, take $\partial S/\partial\psi = 0$, and solve for $\nabla\beta = \vec{B}\times\nabla\psi/|\nabla\psi|^2 + \lambda\nabla\psi$. Its covariant component normal to a flux surface, λ, is the local shear[6] and satisfies $\vec{B}_p\cdot\nabla\lambda = \nabla\cdot(\nabla\psi B_T/R|\nabla\psi|^2)$, with $\vec{B}_p = \nabla\varphi\times\nabla\psi$ the poloidal field.

To proceed further, we adopt a model equilibrium[5,6,13] in which the aspect ratio is large ($r/R \ll 1$), the flux surfaces are shifted circles, and the plasma beta is small but has a finite gradient localized radially in a thin layer. Also, we take $P_{\perp h} = \text{const}$ for $|\tilde{\theta}| \le \theta_0$ and zero elsewhere. For the beta values of interest here, we will put $\sigma = \tau = 1$. Strictly speaking, the sharp $P_{\perp h}$ distribution that we take for convenience would make $\tau < 0$ at $|\tilde{\theta}| = \theta_0$; hence our results should be considered as representative of what would obtain for a slightly smoothed-out distribution. In this model the equilibrium equations can be manipulated to give $\nabla\beta = (q/r)[\hat{\theta} + \hat{r}h(\theta)]$, with

$$h(\theta) = S(\theta - \theta_k) - \alpha_c(\sin\theta - \sin\theta_k) - \tfrac{1}{2}\alpha_h[g(\theta) - g(\theta_k)],$$

and

$$g(\theta) = \begin{cases} \sin\theta - (\tilde{\theta}/\pi)[\sin\theta_0 + (\pi - \theta_0)\cos\theta_0], & 0 \le \tilde{\theta} \le \theta_0, \\ (1 - \tilde{\theta}/\pi)(\sin\theta_0 - \theta_0\cos\theta_0), & \theta_0 \le \tilde{\theta} \le 2\pi - \theta_0, \\ \sin\theta - (\tilde{\theta}/\pi - 2)[\sin\theta_0 + (\pi - \theta_0)\cos\theta_0], & 2\pi - \theta_0 \le \tilde{\theta} \le 2\pi. \end{cases} \tag{7}$$

Here, $S = rq'/q$, $q = rB_T/RB_p$, and $\alpha = -2Rq^2P'/B_T^{\,2}$, with primes for $\partial/\partial r$. Also, θ is now the extended poloidal coordinate of the ballooning representation, with $\tilde{\theta}$ its value modulo 2π, and θ_k (the radial wave number) is a constant between 0 and 2π. The ballooning equation (5) becomes

$$\frac{d}{d\theta}[1 + h^2(\theta)]\frac{d\Phi}{d\theta} + (\alpha_c + \tfrac{1}{2}\alpha_h)D(\theta)\Phi = \tfrac{1}{2}\alpha_h D(\theta)\frac{\int d\theta\,\Phi D(\theta)}{\int d\theta[D(\theta) - \alpha_c/2q^2]} \tag{8}$$

with $D(\theta) = \cos\theta + h(\theta)\sin\theta$.

Figure 1 shows various stability boundaries in shear S and core beta α_c, with the beta of the hot particles α_h and their degree of localization θ_0 as parameters, for $q=2$ and $\theta_k=0$. The two dashed curves show the well-known boundaries for first and second ballooning stability (without hot particles). The dotted lines indicate where drift reversal occurs at zero α_h according to the condition $\overline{\omega}_{dh}(\theta_0)=0$ which is easily expressed in terms of elliptic integrals. Thus, use of the Schwarz inequality limits the validity of our stability analysis to the left of the dotted line for a given θ_0. The solid curves in Fig. 1 are the stability boundaries in the presence of hot particles; at every point on these curves, α_h is chosen to have its maximum value allowed by the condition $\overline{\omega}_{dh}\omega_{*h} > 0$.

Although this procedure underestimates stability, the results in Fig. 1 nevertheless indicate that energetic particles trapped, for example, between $\theta = \pm\pi/4$ are able to stabilize ballooning for shear values up to $S=0.9$ and for core beta values up to and beyond the second stability threshold. As θ_0 increases, the amount of stabilizing energetic plasma that can be introduced also increases, but the drift-nonreversal condition becomes more stringent; hence, optimal stability occurs at the intermediate value of $\theta_0 \approx \pi/4$. For $q=4$ and $\theta_0=\pi/4$, stabilization extends up to $S=1.9$, appropriate near the plasma edge. Also, the value of θ_k was varied, to consider modes peaked off the midplane. With $q=2$ and $\theta_0=\pi/4$, the stability boundary for $\theta_k = 3\pi/8$ (approximately the most unfavorable θ_k value for large α_c) has virtually the same minimum in shear at $S=0.9$ as the curve for $\theta_k=0$, but dips abruptly to $S=0.6$ at the intersection of the second stability and drift reversal boundaries.

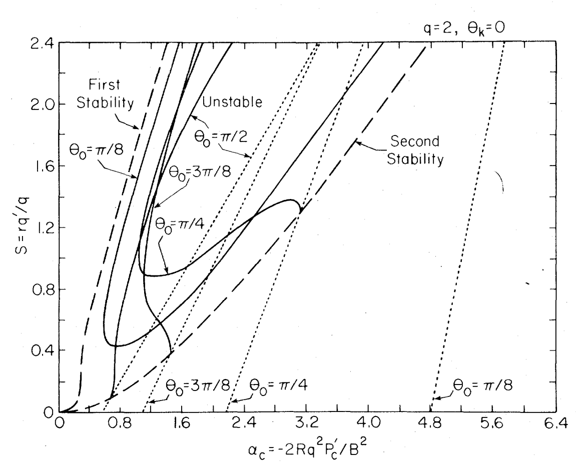

FIG. 1. Marginal stability boundaries in shear S and normalized core beta α_c, for maximal hot beta and various degrees of localization θ_0.

We conclude that it is possible to bridge the ballooning gap between first and second stability by means of energetic particles, which are no longer needed after second stability is attained. Presumably the same scheme could be used in other devices, as has been suggested for the Heliac by Furth and Boozer.[14] The technological requirements for injection or heating of the hot particles, as well as their power balance, deserve further study. Rather high energies are required, as will be seen below. Microinstabilities, such as whistlers or modes near the ion cyclotron frequency, may be possible. On the other hand, finite gyroradius and banana width of the hot particles could improve stability, whereas their slowed-down component could be drift-resonantly destabilizing. Moreover, the same theory as described in this Letter can be applied to "sloshing ions," i.e., with $\overline{\omega}_{dh}\omega_{*h} < 0$, which are found to provide an alternative means for stabilizing a tokamak; although difficult to produce, they do not lead to residual resonant destabilization.

Finally, we note that when the precessional drift of the energetic particles is not large enough to decouple them, marginal stability occurs with a real frequency close to $\overline{\omega}_{dh}$. A simple discussion of finite frequencies can be based on decomposing the energy $\delta W = -\omega^2\delta W_i + \delta W_{fc} + \delta W_{fh} + \delta W_{kh}$ into fluid and kinetic energies for the core and hot species, with δW_i for ion inertia. Let ω'^2 determine the low-frequency ($\omega/\overline{\omega}_{dh} \to 0$) stability as discussed previously, and let $\omega_{\rm mhd}^2 = \omega'^2 - \gamma_h^2$ determine the fluid stability, where γ_h is the growth rate for an unstable flute driven by the hot pressure gradient. The kinetic energy can be approximated for monoenergetic hot particles as

$$\delta W_{kh} \propto \gamma_h^2(\overline{\omega}_{dh}/\omega_{*h})(\omega - \omega_{*h})/(\omega - \overline{\omega}_{dh}).$$

For $\omega, \overline{\omega}_{dh} < \omega_{*h}$, we thus obtain a cubic dispersion relation: $\omega^3 - \omega^2\omega_{dh} - \omega\omega_{\rm mhd}^2 + \omega'^2\overline{\omega}_{dh} = 0$. In the usual MHD limit of small $\overline{\omega}_{dh}$, the expected roots are $\omega^2 = \omega_{\rm mhd}^2$ and a small real root at $\omega = \overline{\omega}_{dh}(\omega'/\omega_{\rm mhd})^2$. In the decoupled hot-plasma limit of large $\overline{\omega}_{dh}$ analyzed earlier in this Letter, we find $\omega^2 = \omega'^2$ and a real root at $\omega = \overline{\omega}_{dh}$. For finite $\overline{\omega}_{dh}$, stability requires

$$[1 + 3(\omega_{\rm mhd}/\overline{\omega}_{dh})^2]^3 > [1 + 9(\omega_{\rm mhd}^2 - 3\omega'^2)/2\overline{\omega}_{dh}^2]^2.$$

Thus the condition $\omega'^2 > 0$ on which our earlier analysis was based is in fact a valid sufficient condition for stability only if $\gamma_h/\overline{\omega}_{dh} < 0.5$, which requires sufficiently hot particles to attain stabilization. For instance, if we estimate $\gamma_h \approx 0.25 \times (N_h T_h / N_i M_i r R)^{1/2}$ applying for $m \geq 2$ and $\beta_{\perp h} \simeq \beta_i$ with T_h and N_h the hot-particle temperature and density, this condition becomes $T_h/T_i > (rR)^{1/2}/2m\rho_i$, where T_i and ρ_i are the plasma ion temperature and gyroradius and m the poloidal mode number. For D-T-reactor-like parameters ($r = 1.5$ m, $R = 5$ m, $B = 5$ T, and $T_i = 10$ keV), $T_h \gtrsim 2.1$ MeV for $m = 2$ is required. We also find that instability sets in at a finite frequency of order $\overline{\omega}_{dh}$. For example, if γ_h^2 is very small, a resonant mode occurs when $\omega_{\mathrm{mhd}}^2 \approx \overline{\omega}_{dh}^2$, with onset at $\omega \simeq \overline{\omega}_{dh}$. A detailed evaluation for the case of internal kinks also predicts a similar onset.[15] This result has been postulated as an explanation for the recent observation on the PDX tokamak of so-called fishbone oscillations, which rotate approximately at the precession rate of the injected beam particles.[16]

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