Electron Delocalization by a Magnetic Field in Two Dimensions

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The problem of two-dimensional localization in the presence of a magnetic field is reconsidered. The existence of extended electronic states is demonstrated by use of the replica formalism and duality arguments. These states are analogs of $\theta = \pi$ vacua in four-dimensional Yang-Mills theories, and occur at the center of each Landau band. The present results complete the explanation of the integrally quantized Hall effect.

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Recently, there has been a great deal of excitement about the two-dimensional electron gas in the presence of a magnetic field. This was sparked by the discovery of the quantized Hall effect.[1] This effect was predicted by Ando, Matsumoto, and Uemura,[2] and later clarified by Laughlin[2] and Halperin,[2] on the basis of the response of electronic states to the presence of magnetic flux.

Their work, as well as related ideas of Thouless,[3] requires that even with the addition of impurity scattering, at least some of the electronic wave functions remain extended. However, as was argued by Abrahams *et al.*[4] and shown more rigorously by Wegner and others,[5] the states of a two-dimensional noninteracting electron gas (the zero-temperature Anderson model) are all strictly localized without a magnetic field. It is therefore extremely important to show the mechanism whereby "conventional" localization breaks down and extended states appear in such a field. In this work, we propose an argument based on topological excitations and 't Hooft[6] duality to demonstrate that extended wave functions do indeed exist, at least at certain special values of the Fermi energy.

We begin by using the standard replica method to study the Green's functions of the Hamiltonian

$$H = (1/2m)(-i\nabla + e\vec{A}/c)^2 + V(r) \equiv \Pi_\mu \Pi^\mu + V(r), \tag{1}$$

where $V(r)$ is a white-noise random potential with variance g. Since H is Hermitian, we can introduce complex Grassmann fields ψ^+, ψ^- and carry out the Gaussian average over $V(r)$. This leads to a generating functional of the form

$$Z = \int DQ \int D\bar{\psi} D\psi \exp L(\bar{\psi}, \psi, Q),$$
$$L = -(1/2g)\sum Q^2 + \psi_a^p(r)[E + is_p\eta - \Pi_\mu \Pi^\mu]\delta_{ab}^{pp'}\psi_b^{p'}(r) - i\bar{\psi}_a^p(r) Q_{ab}^{pp'}\psi_b^{p'}(r), \tag{2}$$

where s_p is $(1,-1)$ and η is a positive infinitesimal. $Q_{ab}^{pp'}$ is a Hermitian matrix residing in U($2n$) with replica indices a, b running from 1 to n and $p = (+,-)$. In order to study localization, we need to focus on the critical fluctuations of the Q fields. In direct analogy with work on the Anderson model,[5] this is done by studying the excitations around the mean-field solution. This leads[7] to an effective nonlinear σ model with the Lagrangian

$$L(\tilde{Q}) = -\tfrac{1}{4}\sigma_{xx}{}^0 \operatorname{Tr} \partial_\mu \tilde{Q}\, \partial^\mu \tilde{Q} - \tfrac{1}{8}\sigma_{xy}{}^0 \operatorname{Tr}\tilde{Q}[\partial_x \tilde{Q}, \partial_y \tilde{Q}], \tag{3}$$

where $\sigma_{\mu\nu}{}^0$ is determined by the underlying short-ranged theory and can be interpreted as the noncritical contribution to the conductivity tensor. In its simplest form, namely in mean-field theory, $\sigma_{xx}{}^0$ is discussed by Ando, Matsumoto, and Uemura.[2] $\tilde{Q}$ belongs to the symmetric space U($2n$)/U(n) $\otimes$ U(n) and may be parametrized as $T^{-1}s_p T$, where $T \in$ U($2n$).

Field theories involving Lagrangians of unitary σ models have previously been suggested by several authors[8] for the magnetic field problem. However, their analyses overlooked the existence of the second term involving $\sigma_{xy}{}^0$ in Eq. (3). Indeed, this term is zero to all orders in perturbation theory and therefore cannot effect the perturbative β function derived by Brezin, Hikami, and Zinn-Justin.[9] In fact, this term is a topological invariant already noted in several related σ models,[10] and is directly analogous to terms of the form $\theta\int d^4x\,\mathrm{Tr}F^{\mu\nu}\tilde{F}_{\mu\nu}$ appearing in four-dimensional gauge theories.[11] We will now proceed to show that this term will drastically alter the scaling behavior of the theory and thereby lead to extended states.

The topological invariant characterized by the expression

$$(\sigma_{xy}{}^0/8)\int d^2x\,\mathrm{Tr}\tilde{Q}[\partial_x\tilde{Q},\partial_y\tilde{Q}]\equiv 2\pi i q\sigma_{xy}{}^0$$

describes the mapping of compactified two-dimensional space to the coset-space manifold. For all n, this mapping is classified by the homotopy group $\Pi_2(\mathrm{U}(2n)/\mathrm{U}(n)\otimes\mathrm{U}(n))=Z$, the set of integers.[12] This immediately implies that the theory is only sensitive to $\theta\equiv 2\pi\sigma_{xy}{}^0 \bmod 2\pi$. It may seem surprising that this topological structure is retained even when n goes to zero. This can be understood by recognizing that $\mathrm{U}(2n)/\mathrm{U}(n)\otimes\mathrm{U}(n)$ is also $\mathrm{SU}(2n)/\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(n))$, where we have divided out the overall phase of T. This still leaves a relative phase between the two $\mathrm{U}(n)$ subspaces, $p=+,-$, of $\mathrm{SU}(2n)$. By a standard result of homotopy theory, $\Pi_2(G/H)=\Pi_1(H)$, if G is simply connected. This implies that the relevant classification may be alternatively described as $\Pi_1(\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(n)))=Z$. Because the Z arises as a result of the U(1) piece, the replica index n is irrelevant. Later, we will extend our boundary conditions to allow for nonintegral topological charge and thereby construct the analogues of 't Hooft's twisted boundary conditions.[6]

To understand the significance of this extra term, one must undertake a nonperturbative analysis of the theory. One approximation, good in relatively weak coupling, is to saturate the functional integral with "instanton" configurations.[13] In our context, instantons are solutions of the classical field equations with nonzero topological charge. The best way to exhibit these solutions is to use the parametrization of MacFarlane,[14] where

$$\tilde{Q}=1-2MM^\dagger,\quad M=\begin{pmatrix}KN\\N\end{pmatrix},\tag{4}$$

where K and N are $n\times n$ matrices which satisfy $(K^\dagger K+1)N^2=1$. It can be shown that the functional form

$$K_{ab}(z=x+iy)=c_a\delta_{b1}\prod_{i=1}^{k}\frac{z-d_a{}^i}{z-e^i}$$

is a k-instanton solution with topological charge $q=k$ and action $4\pi k$. It has a total of $n(k+1)+k$ complex collective coordinates, which reduce at $n=0$ to just k, the instanton positions. These solutions involve a nontrivial winding of the gauge field $A_\mu{}^{ab}=M^\dagger\partial_\mu M$ around the poles occurring at $z=e_i$. These solutions are embeddings of the CP^n $[\mathrm{U}(n)/\mathrm{U}(n-1)\otimes\mathrm{U}(n)]$ model instantons; because only one column of the matrix is nontrivial, these solutions survive in the zero-replica limit.

Generalizing the work of Gross,[15] one can show[16] that these instantons make a nonperturbative contribution to the β function of the form $c\cos(\theta)\exp(-4\pi\sigma_{xx}{}^0)$, where c involves a complicated determinantal computation. At $\theta=0$, this causes σ_{xx}, the true conductivity, to approach zero more rapidly than one would expect on the basis of asymptotic freedom alone, and hence enhances localization. As θ approaches π, the effect of these excitations is in exactly the opposite direction. To the extent that one can trust this crude approximation, the β function would actually have an infrared-stable zero.[17] This would imply the existence of extended states for values of the magnetic field and Fermi energy that give $\sigma_{xy}{}^0$ in this range.

Of course, the dilute-instanton gas argument is rather heuristic. However, we can give a better argument that as θ goes from 0 to 2π, localization must break down somewhere and at least one extended state must appear. To do this, we use 't Hooft's idea[6] of applying a Z_2 twist to the boundary conditions defining the functional integral of the theory. Recall that a mapping of compactified two-dimensional space to the group manifold had to give rise to an integer value of the topological charge. It can easily be verified that this corresponds to a relative phase rotation by 2π. Motivated by the ideas of Edwards and Thouless,[18] we now consider boundary conditions such that there is a relative phase rotation of π. Specifically, consider the boundary conditions on the T field

$$T(x,L)=T(x+L,L),$$

$$T(x,0)=(-1)^kT(x+L,0),$$

where $k=(0,1)$. A typical example of a field with $k=1$ is

$$T(x,t)=\exp\left(\frac{i\pi x\tau_z}{2L}\right)\exp\left(\frac{i\pi t\tau_x}{2L}\right)\exp\left(\frac{i\pi x\tau_z}{2L}\right),$$

where L is the size of the system and the Pauli matrices τ_x, τ_z act on the $+,-$ indices. Note that $\tilde{Q}$ behaves nontrivially as we traverse the sample, explaining the lack of contradiction with the original homotopy argument.

Now, define the free energy $\tilde{F}(e,\theta)$ as the logarithm of the Z_2 transform of $e^{-F(k,\theta)}$, where F is the free energy in the presence of the boundary condition denoted by k:

$$\exp[-\tilde{F}(e,\theta)] = \sum_{k=0,1} e^{ik\pi e+ik\theta/2}\exp[-F(k,\theta)]. \quad (5)$$

The term in the exponent proportional to $k\theta/2$ arises because the $k=1$ boundary conditions above give rise to half-integral topological charge. This definition makes sense because it can be shown[16] that in a Hamiltonian approach to this quantum theory, the label e can be affixed to quantum states by studying their behavior under gauge transformations that do not vanish at spatial infinity. At $\theta\sim 0$, the standard renormalization-group prediction of localization implies that $F(0,\theta)$ will differ from $F(k,\theta)$ by terms of the form $e^{-L/\xi}$, with ξ the localization length. The "duality" definition then requires that $\tilde{F}(e,\theta)\sim L/\xi$. Another way of saying this is that the system is very sensitive to the presence or absence of the "electric" twist e. Now, let us study the role of larger values of θ. For trivial boundary conditions, $\theta=0$ is the same as $\theta=2\pi$. However, in the presence of k twists, θ has the additional effect of shifting e to $e+\theta/2\pi$ [Eq. (5)] (Refs. 19 and 20). Therefore, consider the object $\tilde{F}(-e,\theta)-\tilde{F}(0,\theta)$. At $\theta=0$, it equals $1/\xi$; at $\theta=2\pi$, it equals $-1/\xi$. Clearly, continuity demands that this function be zero somewhere in the range $0\leqslant\theta\leqslant 2\pi$. The vanishing of the difference means that at those particular values of θ, the system becomes insensitive to e and, via duality, is in fact sensitive to nonzero k. This then is a breakdown of localization.

Our argument for extended states is in some sense similar to the ideas of Halperin,[2] in that it only concerns the effect on the system of a complete cycle in θ space, which affects configurations that locally can almost always be thought of as gauge transformations of trivial ones. It leaves open the question of exactly where the extended states appear. The simplest possibility, that only $\theta=\pi$ is in a different phase, would lead to the breakdown of localization at exactly the values of the Fermi energy which correspond to the center of the band for a given magnetic field strength ($\sigma_{xy}{}^0=\frac{1}{2}ne^2/h$). That this may indeed occur is suggested by analogous results in the large-n limit of CP^n models,[21] and would be consistent with similar ideas that arise in other, less systematic, approaches to this problem.[22] However, we cannot at this point rule out the possibility of a band of extended states near $\theta=\pi$.

In summary, we have shown that the presence of $\sigma_{xy}{}^0$, directly attributable to the magnetic field, will cause extended states to appear at or near the center of the Landau bands. These results on the phase structure are consistent with the general arguments of Refs. 2 and 3, and resolve the localization paradox stated at the outset. Using this field-theoretic formalism, one can derive a full theory of the quantized Hall effect which contains explicit demonstrations of the Laughlin approach as well as the results contained here. This will be presented elsewhere.[16]

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