Nonperturbative Weak-Coupling Analysis of the Liouville Quantum Field Theory

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A systematic weak-coupling expansion is developed for the Liouville quantum field theory on a periodic spatial interval. Matrix elements of various Liouville operators are computed to order g^8 by perturbation in nonzero modes about the exact solution of the zero-mode problem. To this order the results agree with the explicit operator solution given previously by Braaten, Curtright, and Thorn, in which the Liouville field was expressed in terms of a free pseudoscalar field by means of an operator Bäcklund transformation.

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Several theoretical physics problems can be reduced to the Liouville field theory in two dimensions,

$$(\partial^2/\partial\sigma^2 - \partial^2/\partial\tau^2)\varphi = (4m^2/g)e^{2g\varphi}. \quad (1)$$

For example, Polyakov[1] reduced the quantization of a relativistic string to a Liouville theory, and in so doing, pointed out possible applications of the theory to statistical mechanics and elementary particle physics.

Last year,[2] Curtright and Thorn presented a conformally covariant quantization of the Liouville theory on a finite spatial interval, $0 \leq \sigma \leq 2\pi$, with periodic boundary conditions (i.e., a circle). They showed that the spectrum of the theory was essentially identical to that of a free massless field on a circle. Later, Braaten, Curtright, and Thorn[3] gave an exact operator solution of the theory, based on a quantum Bäcklund transformation.[4]

However, operator manipulations of the kind used[3] to construct the exact solution often involve mathematical subtleties. Such calculations should be checked with use of perturbation theory. In addition, it is imperative to evaluate correlation functions for the Liouville operators to decide the consistency of Polyakov's string theory, and to extract useful information from it. Correlations for the operator solution in Ref. 3 have been investigated only for weak coupling. Before attempting to evaluate such matrix elements for arbitrary coupling, an independent check of the weak-coupling results is desirable. Furthermore, it has recently been suggested[5] than an expansion in powers of the coupling constant g is not useful for the infinite-volume Liouville theory unless one breaks translational invariance.

In this Letter, we present a systematic weak-coupling analysis of the Liouville theory on a circle which maintains translational invariance. We explain how the above checks can indeed be carried out. We treat the zero modes of the field operators nonperturbatively, and we perturb in nonzero-mode operators coupled weakly to the zero modes. Within this framework, we compute matrix elements of the Liouville operators $e^{\alpha g\varphi}$ between energy eigenstates with energies of order g^2. The computation is done to second order in the nonzero modes. Corrections to our results are of order g^{10}.

Next we compare these perturbation-theory results to the weak-coupling limit of the operator solution of Ref. 3, wherein explicit free-field expressions are given for the operators $\partial_\mu\varphi$, $e^{g\varphi}$, and $e^{2g\varphi}$. To make the comparison, it is convenient to decompose the free-field state space as $\mathcal{H} = \mathcal{H}^{(+)} + \mathcal{H}^{(-)}$, where the subspace $\mathcal{H}^{(\pm)}$ is generated by states of the form

$$\prod_{i=1}^{n} T_{\mu_i\nu_i}{}^{\psi}(x_i)(e^{+iKQ} \pm e^{-iKQ})|0\rangle, \quad K \geq 0. \quad (2)$$

In Eq. (2), $x^\mu = (\tau, \sigma)$, the state $|0\rangle$ is the unique free-field zero-energy eigenstate, $Q \equiv (1/2\pi) \times \int_0^{2\pi} d\sigma\,\psi$, and ψ is the massless free field of the Bäcklund transformation. Also,

$$T_{\mu\nu}{}^{\psi}(x) \equiv :\partial_\mu\psi\partial_\nu\psi: - \tfrac{1}{2}g_{\mu\nu}:(\partial\psi)^2: - (1/2g)(1 + g^2/2\pi)(\epsilon_\mu{}^\lambda\partial_\lambda\partial_\nu + \epsilon_\nu{}^\lambda\partial_\lambda\partial_\mu)\psi, \quad (3)$$

where $\epsilon_0{}^1 = 1$ and the colons denote free-field normal ordering. Note that the form and coefficient of the total derivative term are required by the quantum Bäcklund transformation and conformal invariance.

The perturbation-theory results and the weak-coupling limit of the operator solution agree to the order calculated, i.e., g^8, provided one identifies the Liouville state space with $\mathcal{H}^{(-)}$.

First let us discuss the perturbation-theory approach. The Liouville Hamiltonian is

$$H=\tfrac{1}{2}\int_0^{2\pi} d\sigma:[\dot{\varphi}^2+\varphi'^2+(4m^2/g^2)e^{2g\varphi}]:. \tag{4}$$

This is normal ordered in terms of the raising and lowering operators of the canonical field φ, as discussed in Refs. 2 and 3.

Perturbation theory will be defined with use of free nonzero-mode states and by use of exact solutions for the zero-mode Liouville quantum mechanical problem. Let

$$H=H_0+H_1, \quad H_0=h+R, \tag{5}$$

where

$$h=(1/4\pi)p^2+(4\pi m^2/g^2)e^{2gq}, \tag{6a}$$

$$R=\sum_{n=1}^{\infty}(a_{-n}a_n+b_{-n}b_n). \tag{6b}$$

The complete set of energy eigenstates for H_0 consists of direct products of separate h and R eigenstates. Energy eigenstates of R are the obvious harmonic-oscillator states, while eigenstates of h are easily determined in the q representation (for example, see Ref. 6). One finds

$$h|E\rangle=E|E\rangle, \quad 0<E<\infty, \tag{7a}$$

$$\langle q|E\rangle=(1/\pi)(2k\sinh\pi k)^{1/2}K_{ik}((4\pi m/g^2)e^{gq}), \tag{7b}$$

$$k=(4\pi E/g^2)^{1/2}. \tag{7c}$$

Here K_{ik} is a modified Bessel function.[7] Note that $\langle q|E=0\rangle=0$, i.e., there is no zero-energy state for the Liouville quantum mechanics, as emphasized in Ref. 6. As we shall see below it is an immediate consequence that there is also no zero-energy state for the weak-coupling limit of the Liouville field theory on a circle.

According to Refs. 2 and 3, H and H_0 have identical spectra and the same number of states at a given energy. We shall denote these exact energy eigenstates as $\|E\rangle$, so that

$$H\|E\rangle=E\|E\rangle. \tag{8}$$

Since both $\|E\rangle$ and $|E\rangle$ belong to the continuum, $0<E<\infty$, we may directly relate these states using the Lippmann-Schwinger scattering formalism. Thus

$$\|E\rangle=|E\rangle+\frac{1}{E-H+i\epsilon}H_1|E\rangle=\sum_{n=0}^{\infty}\left(\frac{1}{E-H_0+i\epsilon}H_1\right)^n|E\rangle. \tag{9}$$

The latter series, and the definition of H_0, H_1, and $|E\rangle$, completely define our perturbation theory for the exact Liouville eigenstates. Note that nonzero-mode excitations are allowed for the H_0 eigenstates appearing on the right-hand side of (9). However, such states are degenerate in energy and complicate the formalism. For simplicity in this Letter we consider only H_0 eigenstates $|E\rangle$ with $E<1$, so that $R|E\rangle=0$. Given this restriction, we then identify the exact and zeroth-order energy eigenstates completely by their energy labels.

Also note that one could let $i\epsilon\to-i\epsilon$ in Eq. (9). This would only produce a change of phase in the state $\|E\rangle$ for $E<1$, as may be seen explicitly in our results for matrix elements to follow. This is due to the nondegeneracy of these states. Thus we may adopt the $i\epsilon$ prescription in (9) without loss of information.

By definition, the perturbation in Eq. (9) is

$$H_1=(2m^2/g^2)e^{2gq}\int_0^{2\pi}d\sigma\{\exp[2g\varphi^-(\sigma)]\exp[2g\varphi^+(\sigma)]-1\}, \tag{10}$$

where $\varphi^{-/+}$ are the creation/annihilation nonzero-mode components of φ. Now consider the matrix elements

$$\langle E''\|:e^{\alpha g\varphi}:\|E'\rangle=\sum_{n'',n'}\langle E''|\left(H_1\frac{1}{E''-H_0-i\epsilon}\right)^{n''}:e^{\alpha g\varphi}:\left(\frac{1}{E'-H_0+i\epsilon}H_1\right)^{n'}|E'\rangle. \tag{11}$$

The terms on the right-hand side of (11) may be evaluated by internal insertion of complete sets of H_0 eigenstates. In so doing, it is clear from (10) that one uses matrix elements of exponentials from the Liouville quantum mechanics. These are

$$\langle E''|e^{\alpha gq}|E'\rangle=[1/4\pi^2g\Gamma(\alpha)](g^2/2\pi m)^{\alpha}(k''k'\sinh\pi k''\sinh\pi k')^{1/2}$$
$$\times|\Gamma(\alpha/2+i(k'+k'')/2)\Gamma(\alpha/2+i(k'-k'')/2)|^2. \tag{12}$$

This is valid for $\alpha > 0$. Note that the states are continuum normalized, $\langle E''|E'\rangle = \delta(\sqrt{E''} - \sqrt{E'})/(4\pi)^{1/2}$.

Now Let $g \to 0$ and, to simplify the present discussion, make the further restriction that both E' and E'' in (11) are of order g^2. This limited regime still permits a nontrivial check of the operator results of Ref. 3. In this regime, one can easily establish a lower bound for the power of g appearing in the general n', n'' term on the right-hand side of (11). We find

$$n'', n' \text{ term} = \begin{cases} O(g^{3(n'+n'')}) & \text{if } n'+n'' \text{ is even,} \\ O(g^{1+3(n'+n'')}) & \text{if } n'+n'' \text{ is odd.} \end{cases} \tag{13}$$

This power counting relies on the exponential suppression occurring in the matrix elements (12) unless $k' - k'' = O(1)$.

Thus in the energy regime under consideration, terms with $n'+n'' \geq 3$ are $O(g^{10})$, while terms with $n'+n''=2$ are $O(g^6)$ relative to the $n'+n''=0$ term in (11). It turns out that the terms with $n'+n''=1$ are also $O(g^6)$ for the matrix elements we consider. Explicitly, we find

$$\langle E'' \| :e^{\alpha g\varphi}: \| E'\rangle = \langle E''|e^{\alpha g q}|E'\rangle \{1 - (g^2/4\pi)^3 \zeta_3[(k'^2 - k''^2)^2 + 2\alpha(\alpha-2)(k'^2 + k''^2) + \alpha(\alpha-2)(3\alpha^2 + 2\alpha + 4)/3 + 8i(k'' + k''^3 - k' - k'^3)/3] + O(g^{10})\}, \tag{14}$$

where ζ is Riemann's zeta function. *A priori*, one would expect $O(g^8)$ corrections to the terms with $n'+n''=1$ or 2 on the right-hand side of (11). We have found by explicit calculation that these $O(g^8)$ terms cancel among the five terms involved. Thus the corrections to the results in (14) are $O(g^{10})$.

It should be noted that (14) implies that the operator Liouville equation (1) holds for these matrix elements to $O(g^8)$.

We next compare the results of the foregoing weak-coupling analysis to the weak-coupling limit of the exact operator solution of Ref. 3. In the same weak-coupling limit, we have evaluated matrix elements of the operators $:e^{g\varphi}:$, $:e^{2g\varphi}:$, and $\dot\varphi \pm \varphi'$ expressed as functionals of the free field ψ, between exact energy eigenstates of the form $|P', 0\rangle = \exp(iP'Q)|0\rangle$ [cf. Eq. (2)] with $P' = gk'$, $P'' = gk''$, k' and k'' of $O(1)$. Only the results for the operator $:e^{g\varphi}:$ will be given here. We obtain

$$\langle P'', 0|:e^{g\varphi}:|P', 0\rangle = \frac{g}{8\pi m}\left(\frac{k''}{\tanh\frac{1}{2}\pi k''}\,\frac{k'}{\tanh\frac{1}{2}\pi k'}\right)^{1/2} \frac{1}{\cosh\frac{1}{2}\pi(k'-k'')} \times \left\{1 - \left(\frac{g^2}{4\pi}\right)^3 \zeta_3[(k'^2 - k''^2)^2 - 2(k'^2 + k''^2) - 3] + O(g^{10})\right\}. \tag{15}$$

Comparing with Eq. (14), we find agreement if and only if one takes states belonging to $\mathcal{H}^{(-)}$. That is, define

$$|E, \pm\rangle = \sqrt{2}\begin{pmatrix}\cos\\ \sin\end{pmatrix}[Q(4\pi E)^{1/2}]|0\rangle. \tag{16}$$

Then we have

$$\exp[-i\chi(E'')]\langle E'', -|:e^{g\varphi}:|E', -\rangle \exp[i\chi(E')] = \langle E'' \| :e^{g\varphi}: \| E'\rangle [1 + O(g^{10})], \tag{17}$$

where

$$\chi(E) = \tfrac{8}{3}(g^2/4\pi)^3 \zeta_3 k(1+k^2).$$

We have also calculated matrix elements of $:e^{2g\varphi}:$ to $O(g^6)$, and of $\dot\varphi \pm \varphi'$ to $O(g^8)$. Both agree with (14) on $\mathcal{H}^{(-)}$.

Matrix elements between the states $|E, +\rangle$ do not agree to any order in g with the weak-coupling Liouville theory described above. For $:e^{g\varphi}:$ and $\dot\varphi \pm \varphi'$ the discrepancy on $\mathcal{H}^{(+)}$ only appears in the lowest-order result which occurs as a factor as in (15). In any case, the operator expressions for $\dot\varphi \pm \varphi'$ and $:e^{2g\varphi}:$ satisfy the Liouville equation in the weak-coupling limit to $O(g^6)$ only when the state space is restricted to $\mathcal{H}^{(-)}$.

This clarifies the results of Ref. 3, which indicated that the Liouville equation was satisfied in either the $\mathcal{H}^{(+)}$ or the $\mathcal{H}^{(-)}$ sector. The reason that the operator manipulations of Ref. 3 fail on $\mathcal{H}^{(+)}$ is that the operator expression for $\dot\varphi \pm \varphi'$ was constructed in terms of the inverse of an operator which has, at least in the weak-coupling limit, a zero eigenvector in $\mathcal{H}^{(+)}$. It is not hard to modify $\dot\varphi \pm \varphi'$ in a conformally covariant manner to avoid this problem, but then one can show

that the modified $\dot{\phi} \pm \varphi'$ does not satisfy the Liouville equation on $\mathcal{H}^{(+)}$.

In Ref. 3, it was shown that one could consistently restrict the state space to a subspace of definite zero-mode parity (i.e., $\mathcal{H}^{(+)}$ or $\mathcal{H}^{(-)}$) if and only if the quantity $\nu \equiv \int_0^{2\pi} d\sigma\, \varphi'(\sigma)$ vanished identically. This condition is found to be met for weak coupling to the order calculated, i.e., up to and including terms of order g^8 for matrix elements with E' and E'' both $O(g^2)$.

In conclusion, we have seen that our continuum perturbation-theory analysis confirms the correctness of the operator construction of Ref. 3 on $\mathcal{H}^{(-)}$, at least for weak coupling, and provides an independent and systematic calculational method for the Liouville theory.

We should also mention an earlier study using perturbation theory.[8] However, that earlier perturbation analysis selected both free nonzero and free zero modes as the unperturbed Hamiltonian, and did not incorporate the degeneracy-lifting effects evident in the exact solution of the zero-mode Liouville quantum mechanics.

Details of our calculations and extensions of our results will be presented elsewhere.

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