Spectra of Condensed Modes in Spin-Glasses: Mean Field Theory and Time Dependence

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A mean-field theory for spin-glasses is constructed in terms of coexisting spontaneous symmetry breaking in a whole spectrum of localized eigenstates of the exchange interaction. The spectrum moves with time as the smallest frozen modes begin to disorder, providing a simple physical picture of the universally observed features of the time dependence of the susceptibility. Similar dynamics are expected in disordered ferromagnets.

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Although spin-glass behavior has been observed in a wide variety of superficially different systems, certain features appear to be universal. These include a (slightly rounded) maximum or "cusp" in the susceptibility at a temperature T_g and slow frequency (or time) dependence, $\partial \ln\chi / \partial \ln\omega \approx \text{const}$, below T_g.[1,2] Furthermore, T_g itself is ω dependent,[3] and the cusp sharpens as ω decreases. In this Letter I describe a simple mean-field picture which describes these features naturally.

The theory begins with the formal diagonalization of the exchange J_{ij}, yielding a spectrum containing localized and (probably[4]) extended eigenstates $|\lambda\rangle$. In mean-field theory, we have simply[5]

$$\chi_\lambda = \frac{1/T}{1-(1/T)\lambda}, \tag{1}$$

where λ denotes both the eigenstate label and the eigenvalue of J_{ij}. In a ferromagnet (where the eigenstates are plane waves), the Curie temperature is reached when T first reaches the largest eigenvalue of J_{ij}. At lower T, the instability of all other modes is then suppressed by the presence of the magnetization m_0 in the $k=0$ mode: The free spin $\chi = 1/T$ in (1) is replaced by $(1-m_0^2)/T$, leading to $\chi_\lambda^{-1} > 2(T_c - T)$ just below T_c.

The story is different in a spin-glass. First of all, as has been noted several times,[5-7] one cannot create a stable broken-symmetry state in which the local magnetization varies like the wave function $\langle i|\lambda_{\max}\rangle$, since this eigenstate is localized. (Such a state would amount to broken symmetry in a finite system.) However, here we are not concerned with an equilibrium theory. If the time scale of an experiment is shorter than the relaxation time of this mode, the symmetry is effectively broken. So for now we assume that the experimental probe is fast enough that this is true and proceed with a mean-field description of the symmetry breaking.

Thus we start by allowing a finite magnetization $m(\lambda_{\max})$ (self-consistently determined as in the ferromagnet) in the mode of largest eigenvalue $\lambda_{\max}$. What then happens to the other modes as the temperature is lowered? Are their susceptibilities stabilized by the presence of the order parameter $m(\lambda_{\max})$? At first, at least, the answer is evidently not, since the next soft modes are other localized states centered in other regions of the system, with negligible overlap with $|\lambda_{\max}\rangle$. Thus they can also support similar broken symmetries. As the number of sites where these frozen eigenmodes are sizable grows, however, this cascade of local phase transitions is gradually suppressed, since if the new potentially soft modes strongly overlap one or more of the already condensed ones, the suppression mechanism which operates in the ferromagnet prevents their instabilities.[5] Let us describe this effect approximately by

$$-\partial n/\partial T = \rho(T)\xi^d(T)\,[1-n(T)]. \tag{2}$$

Here $n(T)$ is the number of "frozen sites" at temperature T, $\rho(\lambda)$ is the eigenvalue density, $\xi(\lambda)$ is the localization length, and d is the dimensionality. Equation (2) has the solution

$$n(T) = 1 - \exp\left[-\int_T^{\lambda_{\max}} \rho(\lambda)\xi^d(\lambda)\,d\lambda\right]. \tag{3}$$

These frozen modes then produce an effective Edwards-Anderson order parameter

$$q(T) = \int_T^{\lambda_{\max}} [-\partial n(\lambda)/\partial\lambda]\, q_\lambda(T)\,d\lambda, \tag{4}$$

where $q_\lambda(T)$ is the mean square local magnetization for mode $|\lambda\rangle$ (on the sites where $\langle i|\lambda\rangle$ is sizable). In this mean-field theory, each $q_\lambda(T)$ just has the shape of the square of the spontaneous magnetization for an Ising ferromagnet with $T_c = \lambda$: $q_\lambda(T) = 3(\lambda - T)/\lambda + O(\lambda - T)^2$. Equations (2)–(4) formalize the qualitative remarks of Anderson.[5]

Within these approximations it is then straightforward to calculate q, and thereby $\chi = (1-q)/T$.

The qualitative features of the result are apparent: χ begins to deviate from a Curie law as T falls below λ_{max}. It reaches a maximum somewhere around λ_0, which is defined as the value of T where the exponent in Eq. (3) is equal to -1. [This is the T below which $n(T)$ begins to approach unity.] For $\lambda_0 - T \gg \lambda_{max} - \lambda_0$ (but $\lambda_0 - T$ still $\ll \lambda_0$), $q_\lambda(T)$ can be pulled outside the integral in Eq. (4), giving $q \approx 3(\lambda_0 - T)/\lambda_0$, and so χ varies linearly with T, and $|\partial\chi/\partial T|$ is about twice its value for T above λ_{max}. The peak in χ is rather broad, since ρ and ξ are rather small near λ_{max}.

As in the equilibrium theories of Refs. 6 and 7, the distinction between the spectra of J_{ij} and of the susceptibility χ_{ij} is important here. Explicitly, when we calculate $n(T)$ from Eq. (2), we are trying to count the number of soft modes of χ (weighted by their localization volume). If we were just counting the number of modes of J with eigenvalue $\lambda > T$, the factor $1 - n(T)$ in Eq. (2) would be absent (and the result would have no physical meaning). The spectra of χ and J are different, because of the mode-mode interaction which produces a local molecular field due to the condensed modes.

We have implicitly treated the different condensing modes as independent *of each other* in calculating q. The reason that this is not an absurd approximation is that the growth of the condensed region as T is lowered comes essentially from modes which do *not* significantly overlap previously condensed ones (otherwise they would not be unstable) and are therefore nearly independent of them.

The kind of condensed state we have here is quite different from the sort of broken-symmetry state we normally encounter. We have a microscopic condensate $[O(\xi^d)]$ in a macroscopic number of modes, in contrast to the normal macroscopic condensation in a single mode.

We know, of course, that such a mean-field theory is not a correct description of equilibrium, but it is a reasonable characterization of the effective freezing seen in a very fast experiment. Now what happens when we wait longer? Quite clearly, in this independent-mode approximation, the modes of shortest relaxation time, which are those corresponding to the smallest clusters and therefore those with the largest eigenvalues, will begin to flip over, so that their effective q_λ averages out to zero. As we wait still longer, more and more of them will disorder in this fashion, leaving regions of the material temporarily unfrozen. This then allows some more modes of $\lambda < \lambda_0$, whose instabilities had previously been suppressed, to freeze. In effect, we just have the previous calculation to do again, but with the "melted" fast modes at the top of the spectrum removed. Thus, as time goes on, the band of magnetized modes will move towards lower eigenvalues.

More explicitly, let us suppose that the relaxation time $\tau(\lambda)$ for mode λ ($\lambda > T$) has the form $\tau_0 \exp[\Delta(\lambda)/T]$, where the barrier Δ is proportional to the size and the mean square local magnetization of the mode:

$$\Delta(\lambda) = a\lambda q_\lambda(T)\xi^d(\lambda), \tag{5}$$

with a of order unity. Such a form is obtained in an Ising model (for $\Delta \gg T$) in the independent-mode approximation. Then the condition for a frozen mode to survive melting until time t [$t < \tau(\lambda)$] places both upper [$\lambda_{max}(t)$] and lower [$\lambda_{min}(t)$] limits on the eigenvalues of potentially frozen modes, since ξ gets small at large λ, while $q_\lambda(T) \to 0$ as $\lambda \to T$.

Thus, when we repeat the calculations of $n(T)$, the limits of integration in Eq. (3) become $\lambda_{max}(t)$ and $\lambda_{min}(t)$ instead of λ_{max} and T. Now let us suppose that T is low enough that $\lambda_{min}(t)$ is still well below $\lambda_0(t)$, i.e., that the range of the band of condensed states is essentially controlled by the saturation of Eq. (3), and q is insensitive to $\lambda_{min}(t)$. Then the new $\lambda_0(t)$ will be determined by the new $\lambda_{max}(t)$ in the same way that the old λ_0 was determined by the old λ_{max}. The difference is that $\rho(\lambda)$ and $\xi(\lambda)$ in the range of integration in Eq. (3) are now much larger, and so the interval $\lambda_{max}(t) - \lambda_0(t)$ is smaller. Since the size of this interval controls the degree of rounding of the cusp in χ, this means that the cusp grows sharper as it moves to lower T with increasing t (Fig. 1).

To illustrate the kind of time dependence one finds for $\lambda_{max}(t)$ [and, thereby, for $\chi(t)$] consider the case where the spectrum of J_{ij} has a mobility edge, so that we take $\xi(\lambda) = [\lambda_c/(\lambda - \lambda_c)]^\nu$. Then for T near λ_c, taking $q_\lambda(T) = 3(\lambda - T)/\lambda \approx 3(\lambda - \lambda_c)/\lambda$ in Eq. (5), the condition $\tau(\lambda_{max}) = t$ becomes

$$\lambda_{max}(t) = \lambda_c\{1 + [3a/\ln(t/\tau_0)]^{1/(d\nu - 1)}\}. \tag{6}$$

This approximation is good if $T - \lambda_c \ll \lambda_{max}(t) - \lambda_c$. For higher T, the behavior is not so simple, but Eq. (6) illustrates the fact that χ will quite generally depend on t through $\ln t$, a feature consistent with the general pattern of slow time dependence in many systems. Assuming Eq. (6),

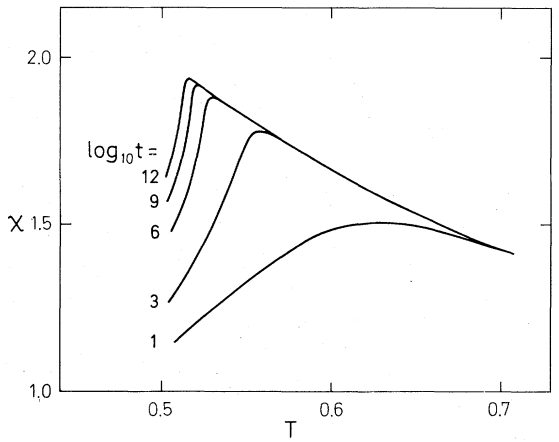

FIG. 1. Calculated susceptibility for several values of $\log_{10} t$, with the assumption of Gaussian eigenvalue density of unit variance, mobility edge $\lambda_c = \frac{1}{2}$, $\xi(\lambda)$ as given in text, and $d\nu = 2$. In these calculations, the sharp cutoffs at $\lambda_{max}(t)$ and $\lambda_{min}(t)$ in the integrations done to get $n(\lambda)$ and $q(T)$ have been replaced by soft ones in which the eigenvalue λ is weighted with a factor $e^{-t/\tau(\lambda)}$. This leads to slightly more rounded maxima than those obtained with sharp cutoffs, but there is no qualitative difference. (t is in units of τ_0.)

we can calculate $\partial \ln \lambda_{max}(t)/\partial \ln t$ and compare it with the quantity $\partial \ln T_g(\omega)/\partial \ln \omega$ measured in experiments. Using $\tau_0 = 10^{-12}$ sec and taking $a = 1$, $d\nu = 2$, I get $\partial \ln \lambda_{max}/\partial \ln t = 0.005$ at $t = 10^{-2}$ sec, which is within a factor of 2 of the values measured by Lundgren, Svedlindh, and Beckman[1,8] and Tholence.[3] Furthermore, this value of $\ln(t/\tau_0)$ corresponds to a value of $\xi(\lambda_{max}(t))$ of four or five lattice spacings, which is consistent, through Eqs. (3) and (4), with the observed degree of rounding of the cusp in χ ($\Delta T_g/T_g$ of a few percent). I also note that the decrease of $\partial \ln \chi/\partial \ln t$ with $\ln t$ is also qualitatively consistent with data on many systems.

For $T > \lambda_c$, $\chi(t)$ eventually approaches a Curie law as all the modes eventually melt, but for $T < \lambda_c$ (if λ_c exists) the spectrum grows very narrow and approaches λ_c as $t \to \infty$, so that the Curie law is never reached.

For $T > \lambda_c$, there is a longest relaxation time in the spectrum, defined by the maximum of $\Delta(\lambda)$ [Eq. (5)]. As $T \to \lambda_c$, this maximum approaches a limit proportional to $(T - \lambda_c)^{1-d\nu}$, and so we find an almost–Vogel-Fulcher law,

$$\tau_{max} = \tau_0 \exp[\text{const}/(T - \lambda_c)^{d\nu - 1}].$$

For $T < \lambda_c$, there is no longest relaxation time; the spectrum of condensed modes just becomes slower and slower forever.

The present picture of the zero-field cooled state as intrinsically time dependent is confirmed in the recent experiments of Lundgren *et al.*,[9] where the time dependence of the spectrum of relaxation times is explicitly measured. They find that the longest relaxation time at a given measurement time after cooling is just the measurement time itself, in agreement with this picture.

The above theory is strictly a mean-field exercise. The only source of the decay of the spin freezing with time is the finite size of the clusters associated with the condensed modes, which have been assumed to flip as rigid, independent units. Excitations like internal fluctuations of the clusters and intercluster interactions have been ignored. It is therefore clear that for long enough times, where sizable overlap between large clusters becomes unavoidable, the present theory breaks down. In particular, it is not trustworthy at equilibrium, where it simply assumes the stability of a condensate with a spatial variation given by the first extended eigenstate of χ.

One can attack this very complicated nonlinear problem by perturbation theory in the mode-mode interactions, as in Refs. 6 and 7. There it was argued, in the context of an approximation valid for large m (number of spin components), that there is no equilibrium phase transition in three dimensions, and the argument is quite suggestive even for $m = 1$ (though the result cannot be said to have been proved except for $m = \infty$). If this is correct for the Ising case, it is apparent that the interactions neglected in the present theory must somehow destroy the mobility edge present in the original spectrum. On the other hand, these interactions do not always hinder a phase transition: In a random ferromagnet (e.g., all $J_{ij} \geq 0$ but with random magnitudes), to which the present analysis ought also to apply, since it only invokes the randomness of J, there is a phase transition, even in two dimensions,[10] where the spectrum of J is completely localized. It seems that in this case the neglected intermode interactions lead to extended states of χ. These problems obviously deserve further study.

However, the inability of the theory to say anything about the existence of an equilibrium phase transition is quite probably irrelevant to a large number of experiments. Apparently, most of the data can be explained in terms of fairly weakly interacting clusters which are not too large, where the present theory is at least a reasonable

starting point.

Finially, it would be interesting to try to observe the kind of slow, "glassy" dynamics inherent in this picture in the random ferromagnet above T_c.

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