Mass Inequalities for Quantum Chromodynamics

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For n-flavor lattice QCD rigorous lower bounds are derived on the masses of baryons, for even $n \geq 6$, and on the masses of flavor-nonsinglet mesons, for even $n \geq 2$. The lower bounds are proportional to the pion mass. The consequence of these inequalities for chiral-symmetry breaking is considered.

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Wilson's Euclidean lattice path integral[1] provides a mathematically unambiguous starting point for a definition of QCD. In the present article I prove, for n-flavor lattice QCD with even $n \geq 2$, the inequality $m_\chi \geq m_\pi$, where m_χ is the mass of any flavor-nonsinglet state produced from the vacuum by a quark-antiquark operator and m_π is the lightest such state with pion quantum numbers. For even $n \geq 4$, I show in addition $m_B \geq (n-3)(n-2)^{-1} m_\pi$, where m_B is the mass of any baryon produced from the vacuum by a three-quark operator.

The inequality on meson masses excludes the possibility that m_π is the threshold for a continuum of multiparticle states incorporating two or more quark-antiquark mesons. For $n \geq 6$ the baryon inequality excludes the possibility that m_π is the threshold energy for a continuum of states incorporating two or more three-quark baryons or antibaryons.

At the price of an additional technical assumption, the proof can be applied without the restriction to even n and without the restriction to $n \geq 6$ for the baryon mass inequality. With this additional assumption the baryon mass inequality becomes the somewhat stronger result $m_B \geq m_\pi$.

Since these inequalities hold uniformly in lattice spacing, lattice volume, and quark mass, they hold also in the infinite-volume continuum limit and in the continuum limit with quark mass taken equal to zero, if these limits exist. If we assume that these limits do exist and that the zero-quark-mass theory has conserved chiral currents, then the baryon mass inequality rules out a Wigner realization of chiral symmetry (in which a baryon becomes massless and pions are massive). If a massive parity doublet is ruled out as the lightest baryon by some additional argument, for example, 't Hooft's anomaly constraint,[2] the present baryon mass inequality implies that the pion mass is zero. Barring a coincidental zero in the matrix elements of the chiral currents between a one pion state and the vacuum, chiral-symmetry breakdown then follows.

The present mass inequalities can also be proved for more general SU(N) gauge theories and may place useful constraints on constituent models of quarks and leptons.

Consider to begin the definition of vacuum expectation values in lattice QCD. The theory is defined, as usual, on a finite-volume hypercubic lattice with gauge variables $U(x,y) \in \mathrm{SU}(3)$ on each oriented nearest-neighbor link and quark fields $\psi_b{}^f(x), \bar{\psi}_b{}^f(x)$ on each site, where f is a flavor index running from 1 to n, and b is a multi-index combining a spin index running from 1 to 4 and a color index running from 1 to 3. Let the lattice spacing be a and the lattice length in direction μ be L_μ. We assume periodic boundary conditions for the gauge fields and antiperiodic boundary conditions for the quark fields. After carrying out an integral over quark fields, the vacuum expectation of a product of quark fields (which is all we need for the present discussion) becomes

$$\langle \prod_i \psi_{b_i}{}^{f_i}(x_i)\bar{\psi}_{c_i}{}^{g_i}(y_i)\rangle = Z^{-1}\int d\mu \, \det(M)^n \det_{ij}[M_{b_i c_i}{}^{-1}(x_i,y_j)\delta_{f_i g_i}] \exp S. \tag{1}$$

Here Z is defined by $\langle 1\rangle = 1$, μ is a product of one copy of Haar measure on each independent link variable, and S is the usual gauge action given by a sum of plaquette contributions,

$$S = g_0{}^{-2} \sum_{(w,x,y,z)} \mathrm{Tr}[U(w,x)U(x,y)U(y,z)U(z,w)], \tag{2}$$

with each nearest-neighbor square counted once in each of its two possible orientations and g_0 equal to

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the bare coupling constant. The quark coupling matrix M in (1) is

$$M(x,y)=(4+m_0a)\delta_{xy}-\tfrac{1}{2}\sum_\mu[(1-\gamma_\mu)U(x,y)\delta_{yx+\hat\mu}+(1+\gamma_\mu)U(x,y)\delta_{yx-\hat\mu}], \tag{3}$$

where m_0 is the bare quark mass, $\hat\mu$ is a unit lattice vector in the $+\mu$ direction, $\{\gamma_\mu\}$ are a set of four Hermitian Euclidean gamma matrices, and spin and color indices have been suppressed for convenience.

Now define a pion field operator

$$\pi^{ij}(x)=\bar\psi^i(x)\gamma^5\psi^j(x), \tag{4}$$

with $i\neq j$. The pion propagator becomes, by Eq. (1),

$$\langle\pi^{ij}(x)\pi^{ji}(y)\rangle=-Z^{-1}\int d\mu\ \det(M)^n\,\mathrm{Tr}[\gamma^5M^{-1}(x,y)\gamma^5M^{-1}(y,x)]\exp S, \tag{5}$$

where the trace in (5) is over color and spin indices which have been suppressed and no sum is intended over i and j. From Eq. (3) we have also

$$[\gamma^5M(x,y)\gamma^5]_{ab}=M_{ba}{}^*(y,x), \tag{6}$$

which is actually a consequence of charge-conjugation invariance for Euclidean QCD. Equations (5) and (6) yield

$$\langle\pi^{ij}(x)\pi^{ji}(y)\rangle=-Z^{-1}\int d\mu\ \det(M)^n\sum_{ab}|M_{ab}{}^{-1}(x,y)|^2\exp S. \tag{7}$$

An argument similar to the derivation of (7) shows that for any quark-antiquark operator and its adjoint,

$$\chi^{fg}=\bar\psi_a{}^f(x)\psi_b{}^g(x)\Gamma_{ab},\quad \bar\chi^{gf}=\bar\psi_a{}^g(x)\psi_b{}^f(x)\bar\Gamma_{ab}, \tag{8}$$

with $f\neq g$, we have

$$\langle\chi^{fg}(x)\bar\chi^{gf}(y)\rangle=-Z^{-1}\int d\mu\ \det(M)^n\,\mathrm{Tr}[\bar\Gamma\gamma^5M^{-1}(x,y)^*\gamma^5\Gamma M^{-1}(x,y)]. \tag{9}$$

Equation (6) implies that $\det(M)$ is real, and thus for even n $\det(M)^n$ is positive. The Cauchy-Schwarz inequality then yields

$$|\langle\chi^{gf}(x)\bar\chi^{fg}(y)\rangle|\leq|\langle\pi^{ij}(x)\pi^{ji}(y)\rangle|. \tag{10}$$

It follows from Lüscher's transfer matrix formulation of lattice QCD,[3] however, that for sufficiently large t and L_4 with $t\ll L_4$,

$$|\langle\pi^{ij}(\vec{x},t)\pi^{ji}(\vec{x},0)\rangle|=\exp[-m_\pi t+o(t)],\quad |\langle\bar\chi^{fg}(\vec{x},t)\chi^{gf}(\vec{x},0)\rangle|=\exp[-m_\chi t+o(t)], \tag{11}$$

where m_χ is the mass of the lightest state produced by $\chi(x)$. Equations (10) and (11) then lead to our first inequality

$$m_\chi\geq m_\pi. \tag{12}$$

By a double application of the Cauchy-Schwarz inequality Eq. (12) can also be proved for particles χ created by field operators $\chi(x)$ with quark and antiquark fields residing at distinct sites and joined by an ordered product of link variables.

Now consider a three-quark baryon field and its adjoint defined by

$$B^{ij}(x)=\psi_a{}^i(x)\psi_b{}^i(x)\psi_c{}^j(x)\Gamma_{abc},\quad \bar B^{ij}(x)=\bar\psi_a{}^i(x)\bar\psi_b{}^i(x)\bar\psi_c{}^j(x)\bar\Gamma_{abc}. \tag{13}$$

Equation (1) then gives the baryon propagator

$$\langle B^{ij}(x)\bar B^{ij}(y)\rangle=-Z^{-1}\int d\mu\ \det(M)^nM_{aa'}{}^{-1}(x,y)M_{bb'}{}^{-1}(x,y)M_{cc'}{}^{-1}(x,y)2\Gamma_{abc}\bar\Gamma_{a'b'c'}\exp S. \tag{14}$$

Since as before for even n, $\det(M)^n$ is positive, Eq. (14) implies, for some positive coefficient γ, the inequality

$$|\langle B^{ij}(x)\bar B^{ij}(y)\rangle|\leq\gamma Z^{-1}\int d\mu\ \det(M)^n[\sum_{ab}|M_{ab}{}^{-1}(x,y)|^2]^{3/2}\exp S. \tag{15}$$

The Hölder inequality can then be used to show that

$$|\langle B^{ij}(x)\bar{B}^{ij}(y)\rangle| \leq \gamma[Z^{-1}\int d\mu\, \det(M)^n \textstyle\sum_{ab}|M_{ab}^{-1}(x,y)|^2 \exp S]^{(n-3)/(n-2)} \times \{Z^{-1}\int d\mu\, \det(M)^n[\textstyle\sum_{ab}|M_{ab}^{-1}(x,y)|^2]^{n/2}\exp S\}^{1/(n-2)}. \quad (16)$$

From an argument similar to the derivation of (7) it now follows that for a product of $n/2$ pion fields we have

$$\langle \pi^{12}(x)\cdots\pi^{(n-1)n}(x)\pi^{21}(x)\cdots\pi^{n(n-1)}(x)\rangle = Z^{-1}\int d\mu\,\det(M)^n[\textstyle\sum_{ab}|M_{ab}^{-1}(x,y)|^2]^{n/2}\exp S. \quad (17)$$

Equations (7), (16), and (17) imply

$$|\langle B^{ij}(x)\bar{B}^{ij}(y)\rangle| \leq \gamma|\langle \pi^{ij}(x)\pi^{ji}(y)\rangle|^{(n-3)/(n-2)}|\langle \pi^{12}(x)\cdots\pi^{(n-1)n}(x)\pi^{21}(y)\cdots\pi^{n(n-1)}(y)\rangle|^{1/(n-2)}. \quad (18)$$

Lüscher's transfer matrix[3] for lattice QCD can then be used to show that for $x\neq y$, the expression $\langle \pi^{12}(x)\cdots\pi^{(n-1)n}(x)\pi^{21}(x)\cdots\pi^{n(n-1)}(x)\rangle$ is a bounded function of x and y. Thus for another positive coefficient γ' we have

$$|\langle B^{ij}(x)\bar{B}^{ij}(y)\rangle| \leq \gamma'|\langle \pi^{ij}(x)\pi^{ji}(y)\rangle|^{(n-3)/(n-2)}. \quad (19)$$

By combining Eq. (19) with Eq. (11) for pions and a corresponding equation for the behavior of the baryon propagator at long distance, we obtain

$$m_B \geq \frac{n-3}{n-2}m_\pi. \quad (20)$$

As was the case for our first bound, Eq. (11), Eq. (20) can also be obtained, by an extra application of the Hölder inequality, for three-quark baryon operators in which the quark fields reside on different sites and are joined by ordered products of link variables.

This completes the proof. A possible variation in the arguments that I have given removes the restriction to $n\geq 6$ for the baryon mass inequality, Eq. (20), and goes as follows:

The matrix M can be written

$$M = m_0 a + R + iI \quad (21)$$

where R and I are self-adjoint. Equation (6) implies that the eigenvalues of I are either zero or occur in matched pairs with opposite sign, and a proof is given by Weingarten and Challifour[4] that R has a nonnegative spectrum. Thus for positive values of $m_0 a$, $\det(M)$ is positive and M is a bounded matrix. Unfortunately weak-coupling expansions[5] show that a continuum limit with either finite or zero quark mass can be obtained only from negative values of $m_0 a$. Weak-coupling expansions and numerical work[6] also suggest, however, that in the infinite-volume limit R is bounded from below by a strictly positive ρ, independent of the gauge field U, and $m_0 a$ of $-\rho$ is the critical point at which the quark mass goes to zero. It is therefore a plausible hypothesis that in the infinite-volume limit, for all positive values of quark mass, $\det(M)$ is positive and the matrix M^{-1} is bounded by a constant independent of U. For staggered fermions[7] R vanishes, finite positive quark mass requires finite positive $m_0 a$, and the required bound on M^{-1} can easily be proved. For staggered fermions, however, other aspects of the derivation of the present inequalities became messier. In any case, with the additional assumption of boundedness for M^{-1} in infinite volume, the Hölder inequality applied to Eqs. (7) and (14) yields for some constant γ''

$$|\langle B^{ij}(x)\bar{B}^{ij}(y)\rangle| \leq \gamma''|\langle \pi^{ij}(x)\pi^{ji}(y)\rangle|, \quad (22)$$

and therefore

$$M_B \geq M_\pi. \quad (23)$$

With the boundedness hypothesis on M^{-1}, Eq. (23) can actually be proved for any multiquark state composed of either quarks, antiquarks, or a combination of the two.

After completion of this work I received two preprints[8] in which estimates related to those discussed here are derived.

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[1]K. Wilson, Phys. Rev. D 10, 2445 (1974) and in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977).

[2]G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum, New York, 1980).

[3]M. Lüscher, Commun. Math. Phys. 54, 283 (1977).

[4]D. Weingarten and J. Challifour, Ann. Phys. (N.Y.) 123, 61 (1977).

[5]N. Kawamoto, Nucl. Phys. B190 [FS], 617 (1981).

[6]D. Weingarten, Phys. Lett. 109B, 57 (1982), and Nucl. Phys. B215 [FS7], 1 (1983); C. Bernard, T. Draper, and K. Olynyk, Phys. Rev. D 27, 227 (1983); R. Gupta and A. Patel, to be published; G. Pawley, D. Wallace, E. Marinari, and F. Rapuano, Nucl. Phys. B220 [FS8], 137 (1983); H. Lipps, G. Martinelli, R. Petronzio, and F. Rapuano, to be published.

[7]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); L. Susskind, Phys. Rev. D 16, 3031 (1977).

[8]C. Vafa and E. Witten, to be published; S. Nussinov, to be published.