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Critical Behavior of the Two-Dimensional Sticks System

I. Balberg,[(a)] N. Binenbaum, and C. H. Anderson
RCA Laboratories, Princeton, New Jersey 08540
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Percolation critical exponents are derived, for the first time, for a two-dimensional system of randomly distributed conducting sticks, which provides a very convenient model for the study of continuum percolation. In the present computer study it was found that the corresponding conductivity exponent, t, has the value of 1.24 ± 0.03 and that the cluster exponents β, γ, and τ have the values 0.14 ± 0.02, 2.3 ± 0.2, and 2.0 ± 0.1, respectively. These results, which are in excellent agreement with values derived for lattices, show that the conductivities of continuum systems and of lattice systems belong to the same universality class.

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In 1974 Pike and Seager[1] considered the percolation problem of a two-dimensional random-sticks system using a Monte Carlo computation. They were able to determine the critical stick length that will bring about the onset of percolation in a system of a given sticks density. Since then, significant progress has been made[2,3] in the understanding of percolation in various systems but no study of the critical behavior of the sticks system has been reported. In particular the resistance of such a system, its critical behavior, and the critical behavior of the stick clusters have not been determined. In addition to being an interesting system in its own right with application for the understanding of resembling composites,[4] the sticks system is a continuum system particularly well suited for computational studies.

Until recently it was not clear whether the continuum percolation belongs to the same universality class as the lattice percolation.[3] Recent comparisons of the cluster-statistics exponents[5-7] have been used to indicate that continuum percolation does belong to the same universality class as lattice percolation. The universality of the conductivity, however, does not necessarily follow the universality of the cluster statistics because of the introduction of Kirchoff's laws in the conductivity problem. The conductivity exponent t has been previously determined for continuum systems by measurements of physical systems[8,9] or by an early computer study of a correlated lattice.[10] The t values derived lead one to expect a universality for the conductivity but this conclusion does not seem to be firm. The reason is that thus far there has been no reported study for which both the cluster exponents and the conductivity exponent were derived simultaneously on a given continuum system, and then shown to be the same as those of lattices. Moreover, the relatively poor accuracy (compared to currently available accuracies of computer studies) of the derived "continuum" t values does not satisfactorily establish their equality to the recently accepted "lattice" t values.[11] Following all of this we have used the sticks system for the determination of the cluster exponents β, γ, and τ, and of the conductivity exponent t. The good agreement between the values obtained here and those derived for lattices and other continuum systems helps in establishing that the conductivities of the lattice and of the continuum belong to the same universality class.

The samples used in the present study were generated by use of the procedure of Pike and Seager.[1] The computer randomly puts N sticks of a given length L in a unit square. Throughout this paper this length is expressed in units of $r_s = 1/(\pi N)^{1/2}$. The orientations of the sticks, θ_i, are chosen randomly and thus the system is expected to be isotropic, i.e., that $P_{\parallel} = \sum_{i=1}^{N} |\sin\theta_i|$ and $P_{\perp} = \sum_{i=1}^{N} |\cos\theta_i|$ are on the average equal. The direction with respect to which θ_i is selected will be called here the longitudinal direction while the perpendicular direction will be called the transverse direction. Even with the large samples used ($N = 1000$) we found slight anisotropies, but they were less than 5% ($0.95 \leq P_{\parallel}/P_{\perp} \leq 1.05$). The intersection between two sticks was determined with use of the criterion of Ref. 1. However, in the present work the clusters formed by the intersections were registered in the following manner. If two sticks intersect, they are given the same cluster number. The cluster numbers are updated with each check of intersection, so that two clusters are given the same cluster number if they have a common stick. When the search for intersections is completed, one obtains the number of the clusters and the number of sticks in each of them. For percolation, one checks the intersection of the sticks with two opposite boundaries. If the boundaries belong to the same cluster, we say that percolation is obtained, and the smallest L which provides percolation is called the critical stick length, L_c.

Once the intersections are registered, a unit resistor is attached to each of them. The sticks themselves are assumed to be resistanceless. The sticks and the boundaries are assumed then to be the junctions or the equipotentials of the resistor network obtained, while the intersections are the resistors of the circuit. In Fig. 1 we illustrate the transformation of a conducting-sticks system into a resistor network. Applying[12] the well-known matrix representation for resistor networks[2] and considering only the percolating clusters, we obtain the resistance $R_{\parallel}$ in the longitudinal direction as well as the resistance $R_{\perp}$ in the transverse direction. Intuitively one expects that taking a prefixed L and a variable N and associating a resistivity to the sticks, rather than a resistance to the intersection, will yield the same critical behavior. We have checked this expectation and found that indeed, within the "experimental" uncertainty, the values of the exponents agree with those obtained in the fixed-N, variable-L, and intersection-resistance case. The reason for presenting here the results of the latter case is that it requires much less computer storage and time, enabling the consideration of larger sticks ensembles ($N = 1000$) and thus narrower error limits in the values derived for the critical exponents.

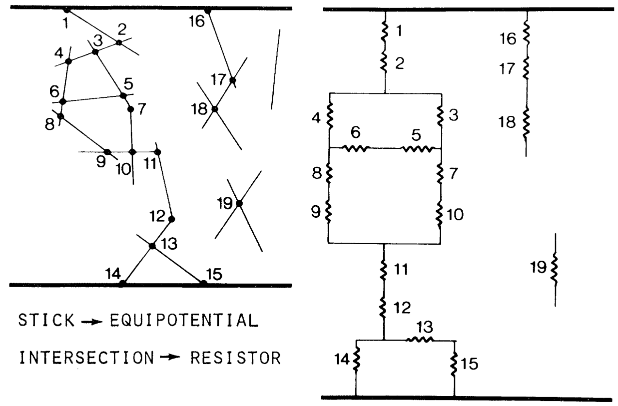

FIG. 1. The transformation of a conducting-sticks system into a resistor network. The sticks are assumed to be conducting and a resistor is assumed to be associated with the intersection of two sticks. Note that the equipotential stick is reduced to a point or a junction in the resistor network.

In the computations we determine the number of sticks in each cluster and the resistance of the sample as a function of L. The present results were derived by use of four "seeds" (or "starters"[3]) which, since we use both directions in this isotropic case, provide eight (essentially independent) sets of data. We have used these sets for the determination of the accuracy of the derived exponents. If we recall that in continuum percolation problems the continuous variable parameter is the density of conducting elements,[5-7] it is expected that in the present problem this parameter is $N/N_c - 1$, where N_c is the critical sticks concentration in the fixed-L, variable-N case. For the reason mentioned above, we have considered the opposite case. This requires, however, the definition of the continuous variable in terms of L. Since it was shown[1] that the onset of percolation is given by $L^2 N_c = L_c^2 N$, we may conclude that $N/N_c - 1 = (L/L_c)^2 - 1$. This point will be further justified by excluded-area arguments.[13]

Following the above considerations, we have presented all the results as a function of $(L/L_c)^2 - 1$. Typical results for the resistance of a sample are shown in Fig. 2. For the eight sets of

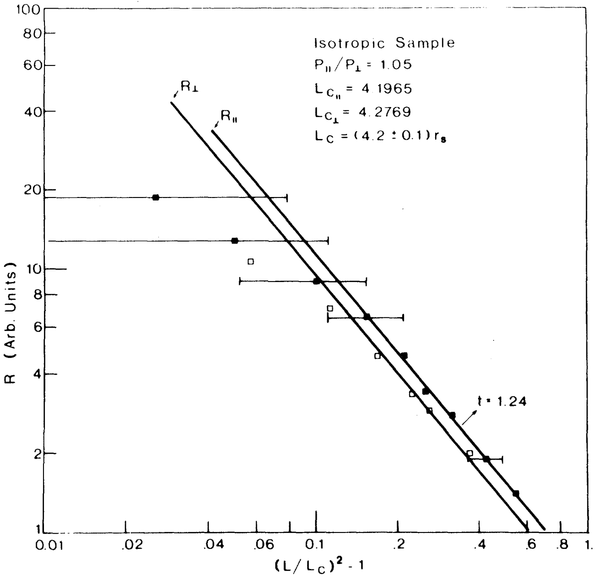

FIG. 2. The longitudinal and transverse resistance of a two-dimensional system of 1000 sticks as a function of the stick-length variable parameter $(L/L_c)^2-1$. These results have been found for one of the seeds. The horizontal "error bars" indicate the regions of the graph to which the results of three other seed (samples) are confined.

data we found that L_c was within the interval 4.2 ± 0.1 which made us conclude that the results for $(L/L_c)^2-1 \lesssim 0.1$ are not to be seriously considered. In fact the rounding of the data towards small values of the variable critical parameter is known to be associated with finite-sample effects. On the other hand, in percolation problems one may expect[14] critical behavior even for the interval 0.1–1 of the variable critical parameter and thus the best fit in this region can yield the correct critical exponent. Indeed, for this region we found, using our nonlinear least-squares-fit procedure,[15] that $t=1.24$ for both directions with one of the seeds. Examining the other three seeds we found that the data points were within the "error bars" shown in Fig. 2. Their least-squares–fit exponents ranged from 1.21 to 1.27. Hence an estimated conductivity critical exponent of $t=1.24\pm0.03$ seems reasonable. This t value is in excellent agreement with the most recent values (1.28 ± 0.03) obtained for two-dimensional lattices.[11,16]

For the determination of the critical exponent β we have considered the onset of the longitudinal percolation in the four seeds. For a large enough sample one may assume[3] that the percolation probability is simply N_p/N, where N_p is the number of sticks which belong to the percolating cluster. Hence we have analyzed the N_p/N data, over the $(L/L_c)^2-1$ range used for the resistance analysis, applying the above nonlinear least-squares fit.[15] Because of the small values of the lines' slopes which describe the corresponding dependence, we carried out the fit to the combined data of all four seeds. We found the best fit to be $\beta=0.14$, and the uncertainty limits[15] to be 0.02. For the determination of γ we have used the same procedure as for the resistance except that the analyzed quantity was $\sum N_s s^2$ and the range considered was $|(L/L_c)^2-1| \gtrsim 0.1$ where $L<L_c$. Here N_s is the number of clusters having s sticks and the summation is over all nonpercolating clusters. The values for the eight sets of data varied between 2.1 and 2.5. A least-squares fit of N_s vs s for $L\approx L_c$ has yielded the value $\tau=2.0\pm0.1$. Again these results are in excellent agreement with the results derived for lattices[3,17] and other continuum systems.[7] Since the cluster statistics of continuum systems has been presented previously,[5-7] we will report details of our cluster-statistics results in a forthcoming publication.

In summary, we have shown that the cluster exponents as well as the conductivity exponent can be derived conveniently by using a two-dimensional system of zero-width sticks. In particular, it seems that the study of the conductivity by computer simulation is easier in the sticks system than in the other commonly studied continuum systems. We have found that in the sticks system both the cluster-statistics exponents and the conductivity exponent are, within presently available accuracies, the same as those derived for lattices.

Hence, this system, being a representative of the continuum percolation problem, indicates that not only the cluster statistics (as has been shown by other researchers) but also the conductivity of the continuum percolation belongs to the same universality class as the lattice percolation.

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[a]Permanent address: The Racah Institute of Physics, The Hebrew University, Jersusalem 91904, Israel.

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