Renormalizability of Quantum Gravity with Cosmological Constant

M. Martellini
Dipartimento di Fisica dell'Università, I-20133 Milano, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, I-27100 Pavia, Italy
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It is shown that quantum gravity with a cosmological constant in a purely affine picture is a power-counting–renormalizable (unitary) theory. The relevance of such a result is based on the equivalence, proved by Kijowski, of the affine formulation of gravity with the standard affine-metric picture.

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The quantization of the gravitational field in the purely metric picture, in which the metric tensor $g_{\mu\nu}$ is regarded as an operator-valued distribution, gives a nonrenormalizable quantum theory.[1] The nonrenormalizability in the metric representation is a consequence of two features: (i) the dimensionless nature of the dynamical variable $g_{\mu\nu}$, and (ii) the presence of the inverse quantities $g^{\mu\nu}$ in the Einstein Lagrangian $(-g)^{1/2}R = \hat{g}^{\alpha\beta}R_{\alpha\beta}$, with $\hat{g}^{\alpha\beta}=(-g)^{1/2}g^{\alpha\beta}$ and $R_{\alpha\beta}$ the Ricci curvature tensor. This makes the Lagrangian an intrinsically nonpolynomial one, when expressed in terms of purely covariant or contravariant components of the metric tensor.[1]

From the physical point of view, however, one can argue that, as implied by the equivalence principle, gravity is associated with a linear connection $\Gamma_{\mu\nu}{}^{\alpha}$. This suggests that gravity is a gauge theory of the diffeomorphism group of the base manifold,[2] much like the theory of the other fundamental interactions. In this picture, the metric of the space-time would play the role of a Higgs field,[2] breaking the structure group of the theory down to its isometry group. The fact also that all gauge theories are renormalizable in terms of the Yang-Mills connection $A_\mu{}^a$ leads us to infer that one of the correct dynamical variables for quantum gravity could be a linear connection $\Gamma_{\mu\nu}{}^{\alpha}$.[3] This approach is also motivated by the fact that a purely affine version of the gravitational interaction is described by a Lagrangian which is the square root of a polynomial.[4] The relevance of such a formulation of gravity is based on its equivalence with the standard metric-affine picture (the so-called first-order or Hilbert-Palatini formalism[5]), which was proved at the classical level in Ref. 4. In this Letter I show, by power-counting arguments, that in the purely affine formalism pure gravity with a cosmological constant is a renormalizable (and unitary) theory.

I start by recalling the basic idea of Ref. 4, where it is shown that the standard metric-affine Lagrangian picture of gravity coincides with the Hamiltonian description (in the sense of the classical mechanics) which one could obtain with a "Hamiltonian" $\mathcal{H}_{MA}(\hat{g}, \Gamma)$ defined by the Legendre transformation of a purely-affine Lagrangian $\mathcal{L}_{PA}(\Gamma, \partial\Gamma) = \mathcal{L}_{PA}(\Gamma, K)$. Let us observe that Kijowski's original formulation[4] concerned a coupled gravity-matter system, but it works also for pure gravity with a nonvanishing cosmological constant. In the following I restrict to the latter case. Therefore $\mathcal{L}_{PA}(\Gamma, K)$ represents only the "purely affine gravitational Lagrangian" constructed only from a symmetric linear connection $\Gamma_{\mu\nu}{}^{\alpha} = \Gamma_{(\mu\nu)}{}^{\alpha}$ and from its first derivatives, and $K_{\mu\nu} = K_{(\mu\nu)}(\Gamma, \partial\Gamma)$ is the symmetric part of the Ricci curvature tensor $R_{\mu\nu}$ of $\Gamma_{\mu\nu}{}^{\alpha}$.[6]

One has

$$\mathcal{H}_{MA}(\hat{g}, \Gamma) = \hat{g}^{\alpha\beta}K_{\alpha\beta}(\hat{g}, \Gamma) + \mathcal{L}_{PA}[\Gamma, K(\hat{g}, \Gamma)], \quad (1a)$$

where $K_{\alpha\beta}$ in Eq. (1a) is now treated as a function of $\hat{g}^{\alpha\beta}$ and $\Gamma_{\mu\nu}{}^{\alpha}$ which one obtains by solving the canonical conjugate momentum definition

$$\hat{g}^{\alpha\beta}(x) = \delta\mathcal{L}_{PA}/\delta K_{\alpha\beta}(x) \quad (1b)$$

with respect to $K_{\alpha\beta}$. Here the "gravitational momentum" $\hat{g}^{\alpha\beta}$, which is a symmetric tensor density of weight $+1$ and of canonical dimension $+2$ (in units with $\hbar = c = 1$), plays the role of the contravariant metric field[7] $k^{-2}(-g)^{1/2}g^{\alpha\beta}$ with $k = (16\pi G)^{1/2}$ the Einstein gravitational coupling constant. Thus one can show[4] that the Hamiltonian equations

$$K_{\alpha\beta} = -\delta\mathcal{H}_{MA}/\delta\hat{g}^{\alpha\beta}, \quad (2a)$$

$$\nabla_\alpha \hat{g}^{\mu\nu} = \delta\mathcal{H}_{MA}/\delta\Gamma_{\mu\nu}{}^{\alpha} \quad (2b)$$

(∇_α stands for the covariant derivative of $\Gamma_{\alpha\beta}{}^{\sigma}$) are completely equivalent to the Euler-Lagrange equations

$$\delta\mathcal{L}_{MA}/\delta\hat{g}^{\alpha\beta} = 0, \quad (3a)$$

$$\delta\mathcal{L}_{MA}/\delta\Gamma_{\mu\nu}{}^{\alpha} = 0 \quad (3b)$$

in which $\mathcal{L}_{MA}(\hat{g},\Gamma)$ is the Hilbert-Palatini Lagrangian[5] of the usual metric-affine picture of gravity. Explicitly, we can take as the purely affine Lagrangian in Eq. (1a) the Eddington one[8]

$$\mathcal{L}_{PA}(\Gamma,K)=(2/\lambda)[-\mathrm{Det}(K_{\mu\nu})]^{1/2},$$
$$K_{\mu\nu}(\Gamma,\partial\Gamma)\equiv R_{(\mu\nu)}(\Gamma,\partial\Gamma)=\partial_\alpha\Gamma_{\mu\nu}{}^{\alpha}-\partial_{(\nu}\Gamma_{\mu)\alpha}{}^{\alpha}+\Gamma_{\rho\sigma}{}^{\sigma}\Gamma_{\mu\nu}{}^{\rho}-\Gamma_{\tau\mu}{}^{\rho}\Gamma_{\nu\rho}{}^{\tau},\quad \Gamma_{\mu\nu}{}^{\alpha}=\Gamma_{(\mu\nu)}{}^{\alpha},\quad \mathrm{Det}(K_{\mu\nu})\neq 0, \tag{4}$$

where λ is a *dimensionless* (in units with $\hbar=c=1$) positive real constant. Obviously, $\mathrm{Det}(K_{\mu\nu})$ is a polynomial in the connection $\Gamma_{\mu\nu}{}^{\alpha}$ and in its first derivatives. The canonical conjugate momentum is

$$\hat{g}^{\mu\nu}=\delta\mathcal{L}_{PA}/\delta K_{\mu\nu}$$
$$=\lambda^{-1}\{-\mathrm{minor}\,K_{\mu\nu}/[-\mathrm{Det}(K_{\mu\nu})]^{1/2}\},$$

where the numerator on the right hand side means the subdeterminant of the member $K_{\mu\nu}$. We may now (uniquely) associate with $\hat{g}^{\mu\nu}$ the two tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ defined by

$$\hat{g}^{\mu\nu}\equiv k^{-2}(-g)^{1/2}g^{\mu\nu},\quad g\equiv\mathrm{Det}(g_{\alpha\beta})=k^{8}\,\mathrm{Det}(\hat{g}^{\alpha\beta}),$$
$$g_{\alpha\beta}g^{\beta\mu}=g_{\beta\alpha}g^{\mu\beta}=\delta_\alpha{}^{\mu}.$$

Then (1b) is equivalent to

$$g^{\mu\nu}=\lambda k^{-2}K^{\mu\nu}, \tag{5a}$$

$$K^{\mu\sigma}K_{\mu\rho}=K^{\sigma\mu}K_{\rho\mu}=\delta_\rho{}^{\sigma}. \tag{5b}$$

Now, the transformation from the purely affine picture to the metric affine one consists in solving Eq. (5a) with respect to $K_{\mu\nu}$. In this way we get

$$K_{\mu\nu}=\lambda k^{-2}g_{\mu\nu} \tag{6}$$

so that the Legendre transformation (1) reads

$$\mathcal{H}_{MA}(\hat{g},\Gamma)=-2\lambda k^{-4}(-g)^{1/2}. \tag{7}$$

At this point, one can easily check that the Hamiltonian equations (2a) and (2b)

$$K_{\alpha\beta}=-\delta\mathcal{H}_{MA}/\delta\hat{g}^{\alpha\beta}=\lambda k^{-4}(-g)^{1/2}\,\hat{g}_{\alpha\beta},$$
$$\nabla_\alpha\hat{g}^{\mu\nu}=\delta\mathcal{H}_{MA}/\delta\Gamma_{\mu\nu}{}^{\alpha}=0$$

are equivalent to the Euler-Lagrange equations generated by a Lagrangian $\mathcal{L}_{MA}$ defined as

$$\mathcal{L}_{MA}(\hat{g},\Gamma)=-k^{-2}(R[g,\Gamma,\partial\Gamma]-2\Lambda)(-g)^{1/2}. \tag{8a}$$

Here I have set

$$R[g,\Gamma,\partial\Gamma]\equiv g^{\alpha\beta}K_{\alpha\beta}(\Gamma,\partial\Gamma), \tag{8b}$$

$$\Lambda\equiv\lambda k^{-2}. \tag{8c}$$

One recognizes immediately that $\mathcal{L}_{MA}$ in Eqs. (8) is Einstein's Lagrangian in first-order form (5) with a cosmological constant Λ. Of course, the usual purely metric picture "â la Einstein" follows from $\mathcal{L}_{MA}$ by a variational principle in the sense of Hilbert-Palatini.[5]

Having shown that classically (4) is general relativity itself with a Λ term, we now prove that, in the affine representation, quantum gravity is a power-counting-renormalizable theory. For this goal we need the (Heisenberg) field equations associated with (4). These turn out to be of the form (see Ref. 4)

$$\nabla_\rho K_{\alpha\beta}=\nabla_\rho D_\sigma\,\Gamma_{\alpha\beta}{}^{\sigma}-\nabla_\rho D_{(\alpha}\Gamma_{\beta)\tau}{}^{\tau}=0,$$
$$\mathrm{Det}(K_{\alpha\beta})\neq 0\rightarrow\Gamma_{\alpha\beta}{}^{\sigma}\neq 0, \tag{9a}$$

where $D_\sigma\equiv\partial_\sigma+\Gamma_\sigma$ and K is required to have the correct signature. Equations (9a) are clearly polynomial in Γ and its first derivatives. In contrast to the nonpolynomiality of the Eddington Lagrangian (4), such a feature suggests that we could develop the quantum theory starting from (9a). So we split Γ into a fixed classical part, $\bar{\Gamma}$, and a quantum part, γ, i.e., we set $\Gamma_{\alpha\beta}{}^{\sigma}=\bar{\Gamma}_{\alpha\beta}{}^{\sigma}+\sqrt{\lambda}\,\gamma_{\alpha\beta}{}^{\sigma}$. Notice that it is always possible to assume that the base manifold M has a metric structure $\bar{g}$ and to choose $\bar{\Gamma}$ to be the Levi-Civita connection of $\bar{g}$.[4] Thus, in the limit of a flat background space-time M, Eqs. (9a) turn out to be of the form

$$\partial_\eta\partial_\lambda\gamma_{\nu\mu}{}^{\lambda}+2\sqrt{\lambda}\,[\gamma_{\chi(\nu}{}^{\sigma}\partial_\eta\gamma_{\mu)\sigma}{}^{\chi}+\gamma_{\eta(\nu}{}^{\sigma}\partial_\chi\gamma_{\mu)\sigma}{}^{\chi}]+2\lambda\gamma_{\eta(\nu}{}^{\sigma}\gamma_{\mu)\rho}{}^{\chi}\gamma_{\chi\sigma}{}^{\rho}=0 \tag{9b}$$

in the gauge condition

$$\gamma_{\alpha\mu}{}^{\alpha}\sim 0. \tag{9c}$$

The gauge condition (9c) has been set to eliminate the four arbitrary gauge freedoms due to the diffeomorphism group of M. However, (9c) does not fix the volume-preserving diffeomorphisms $\delta x^\mu=\tilde{\epsilon}^\mu(x)$ with $\partial_\mu\tilde{\epsilon}^\mu=\mathrm{const}$, so that an other gauge condition is set, namely

$$\partial^{[\alpha}\gamma_{\nu\mu}{}^{\lambda]}\sim 0, \tag{9d}$$

where the index of the derivative is raised with the flat background metric. It follows by Eqs.

(9) that the *large-momentum behavior* of the bare Feynman rules in a typical one-particle irreducible diagram with I internal lines,[9] V_3 three-vertices, V_4 four-vertices, and L loops is given by (up to ghost's contributions)[10] I factors

$$\sim(p^2+i)^{-1}; \tag{10a}$$

V_3 factors

$$\sim(\lambda)^{1/2}\times p; \tag{10b}$$

V_4 factors

$$\sim\lambda; \tag{10c}$$

and L factors

$$\sim p^4. \tag{10d}$$

Note that the asymptotic behavior of the Feynman propagator for the bosonic variable $\gamma_{\mu\nu}{}^{\alpha}$ has in the gauges (9c) and (9d) the correct form to give a unitary theory. Therefore, we do not expect unphysical poles (such as the Lee-Wick ghosts) other than the usual Faddeev-Popov ghosts.[11] Let w_V be the dimension of the interaction monomial (up to coupling constants) attached to the vertex V, that is[12]

$$w_V=n_V+i_V, \tag{11}$$

with n_V (i_V) the number of derivatives (internal incident bosonic lines) at the vertex V; we find that

$$w_{V_3}=4=w_{V_4}. \tag{12}$$

Equation (12) tells us that each gravitational interaction monomial in the purely affine picture has degree $w_V=4$ and hence, according to power-counting arguments, it is of a *renormalizable type.*

The fact that the renormalization properties of quantum gravity depend so dramatically on the parametrization of the fields was suggested also by Salam,[1] who explicitly introduced an "exponential parametrization" of the metric field in order to obtain finite answers. However, one should note that in the present case the reparametrization of the fields (i.e., the passage from the "configurations" $\Gamma_{(\mu\nu)}{}^{\alpha}$ to the "momenta" $\hat{g}^{(\alpha\beta)}$) is performed by a Legendre transformation, while in Ref. 1 it is an algebraic substitution. One may say, as it has been shown, that the metric-affine picture[8] plays the role of the Hamiltonian version of the purely affine theory.[4] Accordingly, one expects that the S matrices of the associated theories become identical, although their Green's functions are not necessarily the same because of differences in the definition of the renormalization conditions. In the path-integral formalism, this fact means that the *unrenormalized* vacuum-to-vacuum amplitude of the affine, $Z_{\rm PA}$, and of the metric-affine, $Z_{\rm MA}$, theories should be equal.[13]

The formal equivalence $Z_{\rm PA}\sim Z_{\rm MA}$ may suggest a prescription to define a "renormalized" quantum gravity in the metric representation. According to the present power-counting argument, $Z_{\rm PA}$ can be renormalized yielding a generating functional $Z_{\rm PA}{}^{(R)}$ for the renormalized quantum theory. Then one could define the standard renormalized metric-affine generating functional $Z_{\rm MA}{}^{(R)}$ to be $Z_{\rm PA}{}^{(R)}$ under the lift to $\mathcal{L}_{\rm PA}{}^{(R)}$ of the Legendre mapping.[1]

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[6]It is worthwhile to notice that in this approach $g^{(\alpha\beta)}$, $\Gamma_{(\mu\nu)}{}^{\alpha}$, and $K_{(\mu\nu)}$ play the role of "gravitational momentum," "gravitational configuration," and "gravitational velocity," respectively, in the sense of classical mechanics.

[7]One should also require that $\hat{g}^{\alpha\beta}$ obtained in this

way is nondegenerate and has the proper signature. On this problem see C. Reina, in Proceedings of Journées Relativistes, Torino, 1983 (to be published).

[8]A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge Univ. Press, Cambridge, England, 1924).

[9]Let us observe that the *free theory* associated with Eqs. (9) when formally $\sqrt{\lambda} \to 0$ is given in the Feynman gauge by $\Box \gamma_{\mu\nu}{}^{\lambda} = 0$.

[10]M. Martellini, Imperial College Report No. ITCP, 82-82/14 (to be published).

[11]One could show that the gauges (9c) and (9d) give rise to "ghosts from ghosts" which decouple from $\gamma_{\mu\nu}{}^{\lambda}$. This leaves only two degrees of freedom representing a massless spin-two field, as one would expect.

[12]C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

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