Baryon Number in Chiral Bag Models

J. Goldstone and R. L. Jaffe

Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 15 August 1983)

When a massless, isospinor Dirac field is confined to a finite region of space by means of a chiral boundary condition parametrized by an angle θ, the baryon number of the vacuum is shown to be $(\chi/2\pi)(\theta-\sin\theta\cos\theta)$ where χ is the Euler characteristic of the bounding surface. Some implications for chiral bag models are discussed.

PACS numbers: 12.35.Ht, 02.40.+m, 11.30.Rd

There is considerable interest in a class of phenomenological, chirally invariant bag models in which the confined quarks are coupled to meson fields at the bag's surface.[1,2] Some time ago, Chodos and Thorn described a spherically symmetric, "hedgehog" solution to an SU(2)×SU(2) chiral bag model in which the isospin index of the meson field points radially: $\vec{\pi}_k(\vec{r})=\hat{r}_k f(r)$. There has been a recent revival of interest in the hedgehog solution[3,4] spurred in part by Witten's proposal[5] that baryons may be described phenomenologically as the solitons of nonlinear chiral models, much in the manner proposed by Skyrme[6] over 20 years ago. It has become apparent that the nature of baryon number must be reexamined in these models: Skyrme and Witten showed that the meson-field configuration carries a topological charge which they propose to interpret as baryon number; Rho and his collaborators and Gross have noted that the quark spectrum in chiral bag models is not symmetric about zero energy and therefore that the quark vacuum can carry baryon number.

In this Letter we give a calculation of the baryon number, N, of the quark vacuum in a slight generalization of the model of Chodos and Thorn. We study the Dirac equation, $i\gamma^\mu\partial_\mu q=0$, inside a static, three-dimensional region, V, bounded by a smooth surface Σ. Σ need not be connected and V need not be simply connected. q is in the fundamental representation of an ungauged SU(2) symmetry group which may be identified with ordinary isospin. On Σ, q obeys the boundary condition

$$-i\vec{\gamma}\cdot\hat{n}_\beta q(\beta)=\exp(i\vec{\tau}\cdot\hat{n}_\beta\gamma_5\theta)q(\beta)\equiv U(\beta)q(\beta), \quad (1)$$

where τ_k, $k=1,2,3$, are the Pauli matrices, and $\hat{n}$ is the unit, exterior normal to Σ. (Chodos and Thorn studied a sphere.) We find that the baryon number depends nontrivially on θ and on the topology of Σ but is otherwise independent of its geometry:

$$N(\theta)=\chi(\theta-\sin\theta\cos\theta)/2\pi, \quad -\pi/2<\theta<\pi/2. \quad (2)$$

Outside the interval $[-\pi/2,\pi/2]$, $N(\theta)$ is given by $N(\theta+\pi)=N(\theta)$. χ is the Euler characteristic of Σ (equal to twice the number of pieces of Σ minus twice the number of handles). Our result disagrees with the result quoted in Ref. 3.

The baryon number of the vacuum is defined by

$$N(\theta)=-\tfrac{1}{2}\lim_{t\to+0}\sum_n\epsilon(E_n)\exp(-t|E_n|), \quad (3)$$

where the sum is over all positive- and negative-energy single-particle eigenstates. This is a regulated version of $\int d^3x\,\tfrac{1}{2}[\psi^\dagger(x),\psi(x)]$. [The mathematicians[7] define a "spectral asymmetry" $\eta(0)$ as the value of $\eta(s)=\sum_n\epsilon(E_n)|E_n|^{-s}$ at $s=0$. If our N is finite, $N=-\tfrac{1}{2}\eta(0)$]. A CT transformation, $q\to i\gamma^0\gamma_5 q$, transforms θ to $-\theta$ and E_n to $-E_n$, and so $N(-\theta)=-N(\theta)$. A discrete chiral transformation $q\to\gamma_5 q$ transforms θ to $\theta+\pi$, so that $N(\theta+\pi)=N(\theta)$.

We study $dN/d\theta$, given by

$$\frac{dN}{d\theta}=\tfrac{1}{2}\lim_{t\to+0}t\sum_n\frac{dE_n}{d\theta}\exp(-t|E_n|) \quad (4)$$

except at those values of θ at which some $E_n=0$, when N jumps by ±1 if E_n changes from $\pm$ to $\mp$.

By use of the Dirac equation and the boundary condition it is not hard to relate $dN/d\theta$ to the limit of a surface integral involving the Euclidean-space Dirac Green's function, $S_\theta(x,x')$ [where $x=(\vec{x},x_4)$, etc.],

$$\frac{dN}{d\theta}=-\tfrac{1}{4}i\lim_{\eta\to0}\eta\int_\Sigma d^2s_\beta\,\mathrm{Tr}\{\vec{\tau}\cdot\hat{n}_\beta\gamma_5\vec{\gamma}^{\mathrm{E}}\cdot\hat{n}_\beta[S_\theta{}^+(\beta,\beta')-S_\theta{}^+(\beta',\beta)]\}. \quad (5)$$

Here $\vec{\beta}$ is a point on Σ; $\beta=(\vec{\beta},\eta)$ and $\beta'=(\vec{\beta},0)$. $S^+(\beta,\beta')$ is the limit of $S_\theta(x,x')$ as $\vec{x}\to\vec{\beta}$ and $\vec{x}'\to\vec{\beta}$ from the interior of Σ. $\vec{\gamma}^{\mathrm{E}}$ and $\gamma_4{}^{\mathrm{E}}$ are related to ordinary γ matrices by $\vec{\gamma}^{\mathrm{E}}=-i\vec{\gamma}$ and $\gamma_4{}^{\mathrm{E}}=\gamma^0$.

Equation (5) can be derived in another, more physical, way. Consider the vacuum expectation value of the baryon-number current, $\langle \vec{j}(x,t)\rangle$, on the surface in the presence of a time-dependent chiral angle $\theta(t)$. Although $\hat{n}_\beta \cdot \langle \vec{j}(\vec{\beta},t)\rangle$ vanishes when calculated naively with use of the boundary condition (1), we find a finite anomalous value proportional to $\dot{\theta}$ when we define $\vec{j}$ carefully using time splitting. Further, as is to be expected for such a short-distance effect, we find a *local* expression for this anomalous current. $\langle \vec{j}(\vec{\beta},t)\rangle$ is related to the Dirac Green's function in the presence of a time-varying θ. With use of the boundary condition, Eq. (1), it is possible to extract an explicit factor of $\dot{\theta}$ from $\hat{n}_\beta \cdot \langle \vec{j}(\vec{\beta},t)\rangle$. If we define $dN/dt = \dot{\theta}\, dN/d\theta$, a short calculation yields Eq. (5).

To evaluate $dN/d\theta$ we employ a reflection expansion of the Green's function $S_\theta(x,x')$,[8]

$$S_\theta(x,x') = S^0(x,x') + \int d^3\alpha\, S^0(x,\alpha)K(\alpha)S^0(\alpha,x') + \int d^3\alpha_1 d^3\alpha_2\, S^0(x,\alpha_1)K(\alpha_1)S^0(\alpha_1,\alpha_2)K(\alpha_2)S^0(\alpha_2,x') + \dots, \tag{6}$$

where $\int d^3\alpha$ denotes $\int_{-\infty}^{\infty} d\alpha_4 \int_\Sigma d^2 s_\alpha$, $S^0(x,x')$ is the free Euclidean Green's function

$$S^0(x,x') = \frac{1}{2\pi^2}\sum_{k=1}^{4} \frac{\gamma_k^{\rm E}(x-x')_k}{(x-x')^4}, \tag{7}$$

and

$$K(\alpha) = U(\alpha) - \vec{\gamma}^{\rm E}\cdot \hat{n}_\alpha. \tag{8}$$

This expansion for S (with $\theta = 0$) is derived and discussed in Ref. 8. The case $\theta \neq 0$ is a straightforward generalization. The integrations in Eq. (6) are singular as $\vec{x}$ or $\vec{x}'$ approaches a point $\vec{\beta}$ on Σ. There are, however, no singularities in Eq. (6) as $\alpha_i \to \alpha_j$ because

$$\lim_{\alpha_i \to \alpha_j} \bar{K}(\alpha_i)K(\alpha_j) = 0$$

where $\bar{K}(\alpha_i) = U^{-1}(\alpha_i) + \vec{\gamma}^{\rm E}\cdot \hat{n}_{\alpha_i}$ results from moving $K(\alpha_i)$ through $S^0(\alpha_i,\alpha_j)$. Indeed, if $\xi(x)$ is generated by a Dirac distribution on Σ,

$$\xi(x) = \int d^3\alpha\, S^0(x,\alpha)\eta(\alpha), \tag{9}$$

then[8]

$$\lim_{\vec{x}\to\vec{\beta}} \xi(x) \equiv \xi^+(\beta) = \tfrac{1}{2}\vec{\gamma}^{\rm E}\cdot \hat{n}_\beta \eta(\beta) + \int d^3\alpha\, S^0(\beta,\alpha)\eta(\alpha). \tag{10}$$

The apparent singularity in the α integration in Eq. (10) is regulated by a principal-value prescription: One excludes a small three-sphere about β, performs the α integration, and then shrinks the sphere to zero.

With use of Eq. (10) it is possible to obtain a reflection expansion for S_θ when $\vec{x}$ and $\vec{x}'$ both lie on Σ:

$$\lim_{\substack{\vec{x}\to\beta \\ \vec{x}'\to\beta'}} S_\theta(x,x') \equiv S_\theta^{+}(\beta,\beta') = \tfrac{1}{4}[1+\vec{\gamma}^{\rm E}\cdot\hat{n}_\beta U(\beta)]S_\theta(\beta,\beta')[3 - U(\beta)\vec{\gamma}^{\rm E}\cdot \hat{n}_\beta], \tag{11}$$

where $S_\theta(\beta,\beta')$ is defined by Eq. (6) with $\vec{x}$ and $\vec{x}'$ set equal to $\vec{\beta}$ and $\vec{\beta}'$ under the α integrals. Substituting into Eq. (5) yields

$$dN/d\theta = -\tfrac{1}{4} i \lim_{\eta\to 0} \eta \int_\Sigma d^2 s_\beta\, {\rm Tr}\{\vec{\tau}\cdot\hat{n}_\beta\gamma_5[\vec{\gamma}\cdot\hat{n}_\beta + U(\beta)][S_\theta(\beta,\beta') - S_\theta(\beta',\beta)]\}. \tag{12}$$

The virtue of the reflection expansion for $S_\theta(\beta,\beta')$ is that successive terms are successively less singular as $\eta \to 0$. In particular it is possible to show that $S^0(\beta,\beta') \sim 1/\eta^3$, $S_\theta^{1}(\beta,\beta') \sim 1/\eta^2$, $S_\theta^{2}(\beta,\beta') \sim 1/\eta$, and $S_\theta^{m}(\beta,\beta') \sim O(1)$ ($m \geq 3$) as $\eta \to 0$. Here $S_\theta^{k}(\beta,\beta')$ denotes the kth reflection, i.e., the kth term in Eq. (6). S^0 and S_θ^{1} do not contribute to $dN/d\theta$: They do not have sufficiently rich Dirac and SU(2) structure to survive the trace operation. *Thus, the entire contribution to $dN/d\theta$ comes from the second reflection.* To evaluate this explicitly we make use of the fact that an $O(1/\eta)$ term in $S_\theta^{2}(\beta,\beta')$ can only arise from that region of integration where α_1 and α_2 approach β or β'. The (smooth) factors $K(\alpha_1)$ and $K(\alpha_2)$ may be expanded about $\vec{\alpha} = \vec{\beta}$. The only term even in θ which survives the SU(2) trace is proportional to $\sin^2\theta\, \hat{n}_\beta \cdot (\hat{n}_{\alpha_1}\times\hat{n}_{\alpha_2})$. The triple scalar product of the normals is proportional to the Gaussian curvature $\kappa(\vec{\beta})$ of Σ at $\vec{\beta}$. The coefficient of $1/\eta$ in $S_\theta^{2}(\beta,\beta')$ can be evaluated by explicitly performing the $\boldsymbol{\alpha}$ integrals, giving

$$\frac{dN}{d\theta} = \frac{1}{2\pi^2}\sin^2\theta \int_\Sigma d^2 s_\beta\, \kappa(\beta) = \frac{1}{\pi}\chi \sin^2\theta \tag{13}$$

by the Gauss-Bonnet formula where χ is the Euler characteristic of Σ.

In fact, the only *local* expression for $\hat{n}\cdot\langle\vec{j}\rangle$ of the correct dimensionality and the correct transformation properties under $SU(2)_L\times SU(2)_R$ and the discrete symmetries is

$$\hat{n}\cdot\langle\vec{j}\rangle d^2s = C\,\mathrm{Tr}M^{-1}\frac{\partial M}{\partial t}\left[M^{-1}\frac{\partial M}{\partial\lambda}, M^{-1}\frac{\partial M}{\partial\mu}\right]d\lambda\, d\mu, \tag{14}$$

where $M=e^{i\vec{\tau}\cdot\hat{n}\theta}$, λ, μ, parametrize the surface, and C is a numerical constant.[9] This gives $\hat{n}\cdot\langle\vec{j}(\beta)\rangle = 4C\sin^2\theta\kappa(\beta)$ which agrees with Eq. (13) if $C=-1/8\pi^2$. C can be determined by noting that for a sphere $\int_0^\pi (dN/d\theta)d\theta = -8\pi^2 C$, while we know that $N(0)=N(\pi)=0$ and that one E_n crosses zero from $-$ to $+$ so that N jumps from $+\frac{1}{2}$ to $-\frac{1}{2}$ at $\theta=\pi/2$.

We can verify that $dN/d\theta=0$ at $\theta=0$ for a sphere using Eq. (4) directly. Let $\vec{L}$ be the orbital angular momentum, $\vec{S}$ the spin, $\vec{T}$ the isospin, $\vec{J}=\vec{L}+\vec{S}$, $\vec{K}=\vec{L}+\vec{S}+\vec{T}$. At $\theta=0$ the energy levels are classified by $|\vec{J}|^2=j(j+1)$, $j=\frac{1}{2},\frac{3}{2},\ldots$, and are $2(2j+1)$-fold degenerate. For $\theta\neq 0$, each such level splits into two, classified by $|\vec{K}|^2=k(k+1)$, $k=j\pm\frac{1}{2}$ $(=0, 1, 2, \ldots)$. We find that the average of $dE/d\theta$ over each set of $2(2j+1)$ degenerate states at $\theta=0$ is zero, and hence that $dN/d\theta=0$. The result of Ref. 3 was based on the view that only $k=0$ contributes to N, and so the source of our disagreement is clear.

Equation (2) follows from Eq. (13) provided that the only jump in N for $0\leq\theta\leq\pi$ is $-\frac{1}{2}\chi$ as θ increases through $\frac{1}{2}\pi$. For $\theta=\pi/2$ and $E=0$ the Dirac equation and boundary condition become very simple,

$$\vec{\sigma}\cdot\vec{p}q=0, \quad \vec{\sigma}\cdot\hat{n}q=\gamma^0\vec{\tau}\cdot\hat{n}q \text{ on } \Sigma, \tag{15}$$

where $\sigma_1=i\gamma^2\gamma^3$, etc. are the Pauli spin matrices. By using a representation with γ^0 and $(\vec{\sigma}+\vec{\tau})^2$ diagonal we can show that there is (i) one solution with $\gamma^0=-1$, $\vec{\sigma}+\vec{\tau}=0$, and q constant; (ii) one solution with $\gamma^0=-1$, $(\vec{\sigma}+\vec{\tau})^2=8$ for every solution of $\nabla\cdot\vec{E}=\nabla\times\vec{E}=0$ with $\hat{n}\times\vec{E}=0$ on Σ; and (iii) one solution with $\gamma^0=+1$, $(\vec{\sigma}+\vec{\tau})^2=8$ for every solution of $\nabla\cdot\vec{v}=\nabla\times\vec{v}=0$ with $\hat{n}\cdot\vec{v}=0$ on Σ. Here $\vec{E}$ and $\vec{v}$ are just q in an appropriate representation. Comparing case (ii) with electrostatics we see that if Σ has n_0 pieces there are n_0-1 solutions, and comparing case (ii) with a fluid-flow problem we see that if Σ has n_1 handles there are n_1 solutions. We can further show that solutions (i) and (ii) have $dE/d\theta>0$ and solutions (iii) have $dE/d\theta<0$. Thus $N(\pi/2+0)-N(\pi/2-0)=n_1-n_0=-\frac{1}{2}\chi$. We can also show that there are no $E=0$ solutions except when $\cos\theta=0$, and our result for $N(\theta)$ follows.

Our result supports the conjecture that the topological charge in the Skyrme model be identified with baryon number. The topological charge of a hedgehog configuration of the meson field surrounding a bag of arbitrary shape can be computed following Skyrme and Witten:

$$\overline{N}(\theta)=-\chi(\theta-\sin\theta\cos\theta)/2\pi. \tag{16}$$

Here $U(\vec{r})=\exp[i\vec{\tau}\cdot\vec{\pi}(\vec{r})/f_\pi]$ is the nonlinear meson field and $\vec{\pi}=0$ at infinity and $\vec{\pi}/f_\pi=\hat{n}\theta$ on the bag's surface.

We are led to the following amusing picture of the transition from the quark-model limit for large bags to the Skyrme limit for vanishingly small bags. Let us follow a spherical bag (radius R) surrounded by a hedgehog meson field configuration $(\vec{\pi}=\pi\hat{r})$ as R decreases adiabatically from very large values $(R\gg 1/f_\pi)$ to very small values $(R\ll 1/f_\pi)$, accomplished, for example, by changing the bag constant B. At large R we place a single quark in the lowest (nondegenerate) bag eigenmode.[10] $\pi(r)$ is determined dynamically[1]: $\pi(R)/f_\pi=\theta(R)$ varies smoothly from zero at large R to $-\pi$ as $R\to 0$. According to Eq. (14), the topological charge $\overline{N}$ varies from zero as $R\to\infty$ to unity as $R\to 0$. The baryon number on the quark fields begins at unity at large R [$N(0)=0$ but a positive-energy mode is occupied]. As R decreases and $\theta\to-\pi$ the baryon number of the quark state decreases adiabatically to zero. When θ passes $-\pi/2$ the baryon number of the vacuum jumps by unity as the newly appeared negative-energy level is filled. In the state we are following it was never empty. The sum of the baryon number inside the bag and the topological charge outside remains unity for all R. The quark configuration inside the bag begins at large R as a filled negative-energy sea plus a single occupied positive-energy mode. As R decreases the occupied mode drops into the sea and, eventually as $R\to 0$, the quark configuration inside the bag is the trivial, symmetric $(\theta=-\pi)$ vacuum.

This collaboration was begun at the Shelter Island II Conference after valuable discussions with A. S. Goldhaber and D. J. Gross. We thank them and the organizers of the conference. We also thank our colleagues in the Center for Theoretical Physics, and M. F. Atiyah, I. M. Singer, and R. B. Melrose for useful conversations and references. One of us (R.L.J.) thanks Oxford University and the Rutherford Appleton Laboratory for hospitality and support during the completion of this work. This work is supported in part through funds provided by the U. S. Department of Energy under Contract No. DE-AC02-76ERO-3069.

[1]A. Chodos and C. B. Thorn, Phys. Rev. D 12, 2733 (1975).

[2]C. G. Callen, R. F. Dashen, and D. J. Gross, Phys. Rev. D 19, 1826 (1979); G. E. Brown and M. Rho, Phys. Lett. 82B, 177 (1979); G. E. Brown, M. Rho, and V. Vento, Phys. Lett. 84B, 383 (1979); R. L. Jaffe, in *Pointlike Structures Inside and Outside the Nucleon, Proceedings of the 1979 Erice Summer School "Ettore Majorana"*, edited by A. Zichichi (Plenum, New York, 1981); S. Theberge, A. W. Thomas, and G. A. Miller, Phys. Rev. D 22, 2838 (1980), and 23, 2106(E) (1981), and Can. J. Phys. 60, 59 (1982); A. W. Thomas, S. Theberge, and G. A. Miller, Phys. Rev. D 24, 216 (1981).

[3]M. Rho, A. S. Goldhaber, and G. E. Brown, Phys. Rev. Lett. 51, 747 (1983).

[4]D. J. Gross, unpublished.

[5]E. Witten, Nucl. Phys. B223, 422, 433 (1983); G. S. Adkins, C. R. Nappi, and E. Witten, to be published. For earlier work see A. P. Balachandran, V. P. Nair, S. G. Rajeev, and A. Stern, Phys. Rev. Lett. 49, 1124 (1982), and Phys. Rev. D 27, 1153 (1983).

[6]T. H. R. Skyrme, Proc. Roy. Soc. London, Ser. A 260, 127 (1961), and Nucl. Phys. 31, 556 (1962).

[7]M. F. Atiyah, V. K. Patodi, and I. M. Singer, Bull. London Math. Soc. 5, 229 (1973), and Proc. Cambridge Philos. Soc. 77, 42 (1975), and 78, 405 (1975), and 79, 71 (1976); P. B. Gilkey and L. Smith, Commun. Pure Appl. Math. 36, 85 (1983). Our problem is a special case of the class considered in this last paper.

[8]T. H. Hansson and R. L. Jaffe, Massachusetts Institute of Technology Report No. MIT-CTP-1070, February 1983 (to be published).

[9]This is presumably the correct result for the more general boundary condition $-i\gamma\cdot nq_- = Mq_+$ where $\gamma_5 q_\pm = \pm q_\pm$ and M is a $U(n)$ matrix. Certainly the reflection expansion argument for θ varying over Σ leads to Eq. (13) with $\sin^2\theta$ under the integral sign.

[10]Color is inessential here and so we set the baryon number of the quark equal to unity.