Low-Dimensional Chaos in a Hydrodynamic System

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(Received 21 July 1983)

Evidence is presented for low-dimensional strange attractors in Couette-Taylor flow data. Computations of the largest Lyapunov exponent and metric entropy show that the system displays sensitive dependence on initial conditions. Although the phase space is very high dimensional, analysis of experimental data shows that motion is restricted to an attractor of dimension 5 for Reynolds numbers up to 30% above the onset of chaos. The Lyapunov exponent, entropy, and dimension all generally increase with Reynolds number.

PACS numbers: 47.25.-c, 03.40.Gc, 05.70.Ln

Lorenz[1] and Ruelle and Takens[2] have suggested that the onset of fluid turbulence can be described by strange (chaotic) attractors, that is, nonperiodic motion generated by finite-dimensional deterministic dynamics. This contrasts with Landau's suggestion that turbulence is multiperiodic motion with many incommensurate frequencies. An experiment by Gollub and Swinney[3] showed that the Landau hypothesis fails, but until now the strange-attractor hypothesis has eluded verification. Using improved techniques for experimental data acquisition and analysis, we present evidence in this paper that the transition to turbulence in Couette-Taylor flow is initiated by low-dimensional strange attractors.

A deterministic system is chaotic if nearby points in phase space separate at an exponential rate (on the average). This sensitive dependence on initial conditions is reflected by a positive largest Lyapunov exponent[4] λ_1 and positive metric entropy[5] h_μ. The dimension[6] d_μ of an attractor, if small and nonintegral, confirms that the dynamics admits a low-dimensional deterministic mathematical description characterized by a strange attractor. We now describe the experiment and calculations of λ_1, h_μ, and d_μ.

Measurements were made on a concentric cylinder system with radius ratio 0.875, outer radius 5.946 cm, a fluid height-to-gap ratio 20, and rigid stationary end boundaries.[7] The modulated wavy-vortex flow state studied had sixteen Taylor vortices and four azimuthal waves in each traveling wave train.[7] Measurements were made for Reynolds numbers R in the range $10R_c$ to $15R_c$, where R is proportional to the angular velocity of the inner cylinder and R_c is the critical Reynolds number for the onset of Taylor vortex flow. The radial component of the velocity, $V(t_k)$ [where $t_k = k\Delta t$, $k = 1, \ldots, 32\,768$; typically $\Delta t = 6$ ms], was determined by Doppler velocimetry. Using a pulse correlator, we obtained velocity values far more accurate than those obtainable by the usual analog velocimetry methods.

Phase portraits of dimension m can be constructed from the vectors $\{V(t_k), V(t_k+\tau), \ldots, V(t_k+(m-1)\tau)\}$, where τ is essentially arbitrary.[8] Figure 1(a) shows phase portraits at $R/R_c = 10.1$, where the velocity power spectrum contains only sharp peaks at two fundamental frequencies and their combinations, and at $R/R_c = 12.0$ and 15.2, where the spectrum contains broadband noise in addition to the sharp spectral components. Figure 1(b) shows two-dimensional Poincaré sections given by the intersection of orbits in three-dimensional portraits with planes. The closed loop corresponding to the surface of a torus is well defined at $R/R_c = 10.1$; the small amount of scatter presumably arises from instrumental noise. The surface of a torus is still clear, although fuzzier at $R/R_c = 12.0$. However, at $R/R_c = 1.52$ a torus is no longer apparent—phase portraits and Poincaré sections no longer yield useful information. Therefore, we turn to more quantitative methods of data analysis, i.e., com-

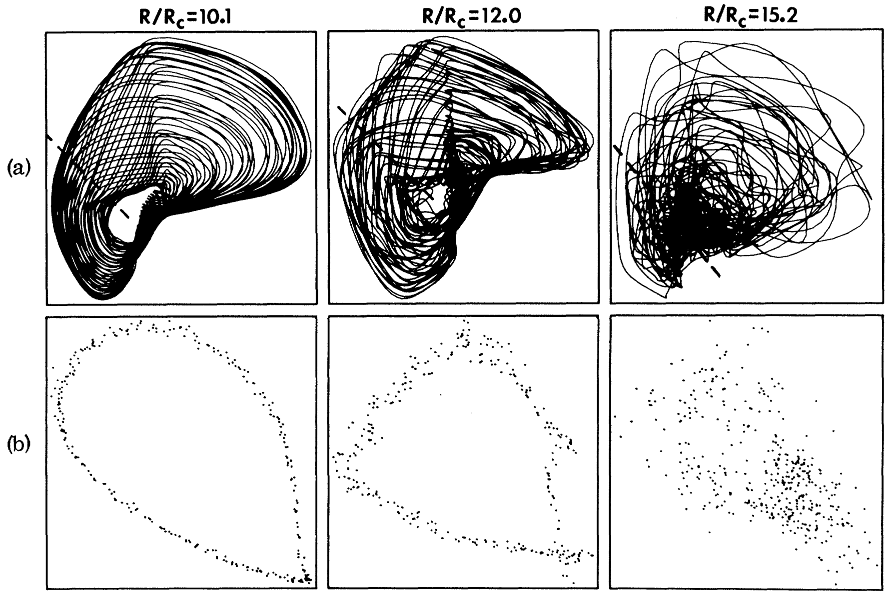

FIG. 1. (a) Two-dimensional phase portraits, $V(t_k+\tau)$ vs $V(t_k)$, where $\tau = 130$ ms. (b) Poincaré sections given by the intersection of orbits in a three-dimensional phase portrait [with the axis normal to the paper given by $V(t_k+2\tau)$] with a plane normal to the paper passing through the dashed line in (a).

putation of λ_1, h_μ, and d_μ.

In chaotic motion nearby orbits diverge at an exponential rate given asymptotically by a positive largest Lyapunov exponent λ_1; in contrast, for multiperiodic motion, $\lambda_1 = 0$. We have developed an algorithm for estimating the nonnegative exponents of an attractor from measurements of a single observable.[9] To find λ_1 we first construct a phase portrait of sufficiently high dimension. We then continuously monitor the long-term evolution of the separation between a pair of initially adjacent data points. When this separation is no longer small, the second point of the pair is replaced by a "nearest neighbor" of the first, subject to the condition that the orientation of the separation vector is most nearly preserved. The average rate of growth of the logarithm of this separation is then our estimate of λ_1. Using files of ~ 300 orbits (~ 100 points/orbit) in five-dimensional reconstructions of the attractors, we found that λ_1 was close to zero before the transition, and generally increased with R after the transition, as shown in Fig. 2. Although our method works well on a variety of model systems with known Lyapunov exponents,[9] we find that for laboratory data a variety of problems cause a dependence of λ_1 on the embedding dimension m. Our interest here, however, is in the behavior of λ_1 with R rather than its precise magnitude, and this is independent of m.

The metric entropy h_μ is the average information gained with each measurement on a dynamical system. For chaotic motion, $0 < h_\mu < \infty$; for multiperiodic motion, $h_\mu = 0$. The metric entropy is believed to be equal to the sum of the positive Lyapunov exponents. To compute h_μ the phase space is partitioned into cells that represent possible outcomes of measurements made with

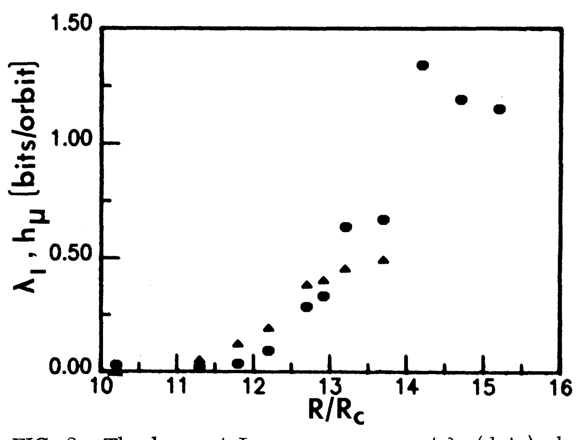

FIG. 2. The largest Lyapunov exponent λ_1 (dots) obtained from five-dimensional phase portraits and the metric entropy h_μ (triangles) as a function of R. The units are bits per orbit (i.e., bits per intersection with a Poincaré section). For calculations of λ_1 and h_μ the data were low-pass filtered with a cutoff at approximately 3 times the higher fundamental frequency.

finite precision.[5] As a trajectory traverses the phase space, it moves through different cells, generating a sequence of measurement outcomes. The probability of occurrence of each sequence of finite length can then be approximated by the relative number of times that it occurs, i.e., $p(S_n) = N(S_n)/N_t$, where $N(S_n)$ is the number of times a particular sequence S_n occurs and N_t is the total number of occurrences of all possible sequences of length n. The average information contained in sequences of length n is $I_n = -\sum_{S_n} p(S_n)\log_2 p(S_n)$ and the metric entropy is the amount of new information, $h_\mu = \lim_{n\to\infty}\lim_{N_t\to\infty}(I_{n+1} - I_n)$. We compute h_μ only at small R where two-dimensional Poincaré maps can be used to obtain accurate values of the entropy. For each value of h_μ in Fig. 2 we estimate an error of ± 0.05 bit/orbit. In addition our technique is known to underestimate h_μ.

The dimension of an attractor provides a way of quantifying the number of relevant degrees of freedom present in dynamical motion. We use three methods[10,11] for computing the dimension of the attractors obtained from our data. The basic idea behind these methods is that the number of points N of a d_μ-dimensional attractor inside an m-dimensional ball of radius ϵ ($d_\mu \leq m$) scales as ϵ^{d_μ}. Our first method of estimating dimension is to compute the average of $\ln N$ for many balls of radius ϵ, and then the slope of $\ln N$ vs $\ln\epsilon$ is determined for increasing m. The second method is similar, except that N is made the independent variable: The distance ϵ from a given point to its Nth nearest neighbor is computed, and then ϵ is averaged over many points, for increasing m. These two methods in principle produce the same number.[6] Our third method, due to Grassberger and Procaccia, is to compute a lower bound on d_μ as described in Ref. 11.

Determination of d_μ by method 2 is illustrated in Fig. 3(a). Plots of $\ln N$ vs $\ln\epsilon$ are approximately straight lines. As m increases the slope increases but approaches an asymptotic value for large m, as illustrated in the inset in Fig. 3(a); confirming that the nonperiodic motion of the fluid takes place on a finite- (in fact low-) dimensional strange attractor. Figure 3(b) shows the growth of d_μ with R.

Finally, we need to consider whether our calculations of λ_1, h_μ, and d_μ can truly distinguish between deterministic chaos and the effects of extrinsic noise. Results for the Couette system were compared to those obtained for a multiperiodic time series with increasing amounts of added noise. λ_1 and h_μ increased with noise level, as expected. However, in contrast to results for the experimental data, these values showed a strong sensitivity to the cutoff frequency of a low-pass filter applied to the test signal. Calculations of d_μ for multiperiodic data with added noise showed no tendency to converge with increased embedding dimension, again in sharp contrast to calculations on the experimental data. We therefore conclude that although there is undoubtedly extrinsic noise in the Couette system, the motion is dominated by deterministic chaos.

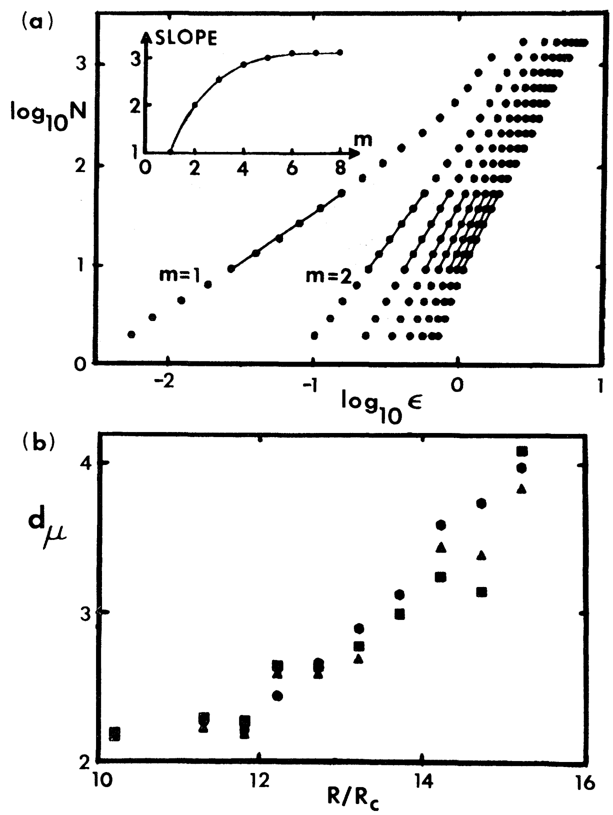

FIG. 3. (a) Curves used to deduce the dimension d_μ by method 2 from data (spanning about 300 orbits) at $R/R_c = 13.7$ for different m. The asymptotic slope for large m (see inset) is an estimate of d_μ. There are systematic errors for large and small ϵ; therefore, as shown in the figure, only the middle portion of the curve is used for the fit. We estimate our error bars to be ± 0.3 at $R/R_c = 10.1$ and ± 0.8 at $R/R_c = 15.2$. (b) The R dependence of d_μ computed with use of methods 1 (dots), 2 (squares), and 3 (triangles), as described in the text.

In summary, we would like to emphasize not the precise values of the λ_1, h_μ, and d_μ, but that above the onset of chaos (marked by the appearance of broadband spectral noise) λ_1 and h_μ become positive and d_μ remains small. The growth of λ_1 and h_μ with R indicates an increase in the unpredictability of the flow and the growth of d_μ with R indicates an increase in the number of active degrees of freedom in the fluid. Although the fluid could potentially have a very large number of degrees of freedom, our studies indicate that there are only a few *relevant* degrees of freedom, certainly less than 5, even at a Reynolds number 30% above the onset of chaos.

While this paper was in preparation we learned of related studies of dimension by B. Malraison, P. Atten, P. Bergé, and M. Dubois, and by J. Guckenheimer and G. Buzyna.[12] The assistance of Mark Haye in efficiently programming the calculations of λ_1 is gratefully acknowledged. Our experiments were conducted at the University of Texas with the support of National Science Foundation Grant No. MEA82-06889.

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