Normalizable Resonance Wave Function, Analyticity, and Decay Law

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The authors show how to construct a normalizable resonance wave function produced by a sharp-cutoff two-body potential using analyticity and basic quantum mechanical requirements. To this end they propose a saturation scheme, and estimate the duration of validity of the exponential decay law. The viability of the model is tested in a deuteron stripping to unbound-state calculation.

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The concept of a resonance occurs very widely in elastic, inelastic, stripping, and rearrangement reactions in many areas of physics. Although the properties of transition amplitudes containing resonance poles are well understood and widely used, a completely satisfactory prescription for the resonance *wave function* is not apparently available despite half a century of work.[1-5] Indeed the asymptotic behavior of the Gamow wave function,[6] which is an eigenstate of the Hamiltonian, gives rise to a probability density which blows up exponentially at infinity. The several variants[1-3] of the eigenfunction theories likewise suffer from shortcomings which limit their usefulness; e.g., the modified (Hartree-Fock) Hamiltonian of Nicolaides and Beck[3] does not have a normalizable eigenstate with positive energy and hence their procedure is not applicable to the two-body problem considered below. Similarly, among the class of wave-packet models[4,5] in which the resonant state $|\xi_j\rangle$ is *not* an eigenstate of the Hamiltonian but a superposition of scattering states $|\psi_k{}^-\rangle$ at $t=0$,

$$|\xi_j\rangle=(2\pi^2)^{-1}\int_0^\infty dk\,k^2|\psi_k{}^-\rangle\rho_j{}^+(k), \qquad (1)$$

the existing models for ρ are unsatisfactory although they lead to normalizable resonant states. Specifically Fonda, Ghirardi, and Rimini's[4] state gives *double* poles in the saturated T matrix at all the dynamical points because of the incorrect appearance of the Jost function $D^+(k)$ in the denominator of ρ and neither Fonda, Ghirardi, and Rimini[4] nor Schlessinger and Payne[5] have any mechanism to fix the arbitrary numerator of ρ.

The need for a satisfactory resonance wave function is crucial in certain areas of application, e.g., stripping to unbound states in nuclear physics where mere knowledge of the T matrix is not adequate. For a reaction $A(d,p)B^*$, $B^*\to n+A$ where A is a spin-zero nucleus and B^* the residual nucleus in a specific resonant state, the outgoing proton angular distribution $\sigma_p(\theta)$ can be related through the distorted-wave Born approximation (DWBA) to a matrix element which explicitly contains $|\chi_n\rangle$, the neutron wave function in B^*. The usual artifacts employed for $|\chi_n\rangle$ are the n-A scattering state subjected to a Gaussian damping[7] or to a contour rotation,[8,9] a quasiparticle state,[2] or a weakly bound state.[10] All these are not really resonant states and hence unsatisfactory choices for $|\chi_n\rangle$; instead, as Schlessinger and Payne[5] point out, a normalizable wave-packet state given by Eq. (1) should describe the final neutron in the continuum. It should be borne in mind that this normalizable state is not a bound state but rather the precise probability of finding a continuum state in it is $|\rho|^2$. As mentioned earlier, however, the existing wave-packet models[4,5] are unsatisfactory and in the present Letter we formulate a model for a two-body resonance wave function produced by a real sharp-cutoff potential using in great detail the powerful tools of analyticity in potential scattering and some basic quantum mechanical constraints. This results in a proper and minimally unique formulation for a resonant state. We shall also see that in applications it gives considerably improved results. We believe that the present method of constructing resonant states has considerable potential for use in several areas of physics.

Consider a resonance j created by a real short-range potential V in a given partial wave l. It can be located through the zero of $D^+(k)$ lying in the fourth quadrant of the complex momentum plane at $k_j=\alpha-i\gamma$, i.e., at $E_j=k_j{}^2/2\mu=E_\alpha-i\Gamma/2$ in the second sheet of the complex energy plane. The Gamow state[6] is then defined as

$$|\psi_j\rangle=(-i\mu k/\pi)a_j(k_j)\lim_{k\to k_j}|\psi_k{}^-\rangle$$

and $a_j{}^2(k_j)\approx 2\pi\gamma/\mu^2$ for narrow resonances. The

vertex function[11] associated with $|\psi_j\rangle$ is given by

$$a_j(p) = \langle p|V|\psi_j\rangle = (-i\mu k_j/\pi)a_j(k_j)\lim_{k\to k_j} T^-(p,k)$$

in terms of a conjugate half-shell T matrix. Instead of $|\psi_j\rangle$ we use Eq. (1) to define unstable state of narrow resonance j and determine ρ as described hereafter.

We first make $|\xi_j\rangle$ equal to its time-reversed state $|\tilde{\xi}_j\rangle$ to ensure that a single real wave function $\xi_j(r)$ alone will describe together the resonance pole at k_j and capture pole at $-k_j^*$. If we impose then the plausibility conditions (i) that the model vertex function $\Omega_j(p) = \langle p|V|\xi_j\rangle$ be approximately equal to $a_j(p)$, (ii) that the matrix elements $\langle\xi_j|H^m|\xi_j\rangle$ be finite for $m = 0, 1, 2$, and (iii) that the resonance saturated half-shell amplitude

$$R_j^+(p,k) = \langle p|V|\xi_j\rangle\langle\xi_j|\psi_k^+\rangle = \Omega_j(p)\rho_j^+(k)$$

possess approximately the same pole residues at $k = k_j, -k_j^*$ as exact $T^+(p,k)$ and no poles at positions of other resonances and bound states, we can verify the following.

Proposition: For a sharp-cutoff potential of range d [i.e., $V(r)\equiv 0$, $r>d$] the minimal form of the expansion coefficient consistent with the plausibility conditions is

$$\rho_j^+(k) = X_j(k) + (-)^l S^+(k)X_j(-k), \qquad (2a)$$

where

$$X_j(k) = D^-(k)F(k)Y(k)/(k^2-k_j^2)(k^2-k_j^{*2}),$$

$$F(k) = (ik)^{-1}\exp[-i(kd - l\pi/2)], \quad Y(k) = C_0 + iC_1k.$$

The *real* constants C_0 and C_1 are obtained from the relation

$$C_0 + iC_1\alpha = -2i\mu\alpha\gamma a_j(\alpha)/[F(\alpha)D^-(\alpha)]. \qquad (2b)$$

The significance of condition (i) is that the wave-packet state and the Gamow state couple with approximately equal strengths to the constituents in a state of relative momentum p. The bound states have been dropped in Eq. (1) to eliminate their contributions to $\Omega_j(p)$. The condition (iii) ensures a close correspondence with standard Breit-Wigner amplitude. These two conditions together with the finiteness of $\langle H^2\rangle$ are the new features in our construction of $|\xi\rangle$. Further we point out that condition (ii) is satisfied in a minimal way by the linear form of $Y(k)$ in Eq. (2). Moreover, many integrals can be done in closed form while computing $\Omega_j(r)$ or $\xi_j(r)$ and also the roles of resonance and capture poles have become symmetrical in Eq. (2a). To our knowledge all these features together could not be achieved in any earlier wave-packet approach.[4,5]

It is now possible to compute the radial wave function $\xi_j(r)$ explicitly with use of Eqs. (1) and (2). Because of space limitations we only mention that $\xi_j(r) \approx \mathrm{Re}\psi_j(r)$ for $r<d$, so that it is regular at $r=0$ like r^l. For $r\geq 3d$, $l\leq 2$, $\xi_j(r)$ behaves like $r^{-(l+1)}$. Both of these are desirable features for our purpose of applications. Further the time-dependent wave function $\xi_j(r,t)\equiv\langle r|\times e^{-iHt}|\xi_j\rangle$ at a fixed large r and for $t>0$ has been determined by the steepest descent technique, and also the outgoing flux across a large sphere of radius r. This results in an estimate for n_j, the number of lifetimes for which the classical exponential law for yield is valid; n_j satisfies $\exp(n_j)/n_j \approx \pi\epsilon^2 E_\alpha/\Gamma$ with $|\epsilon|\ll 1$. We stress that in Refs. 3 and 4 only the survival probability $|\langle\xi_j|\xi_j(t)\rangle|^2$ was considered and not the evaluation of $\xi_j(r,t)$ and the flux.

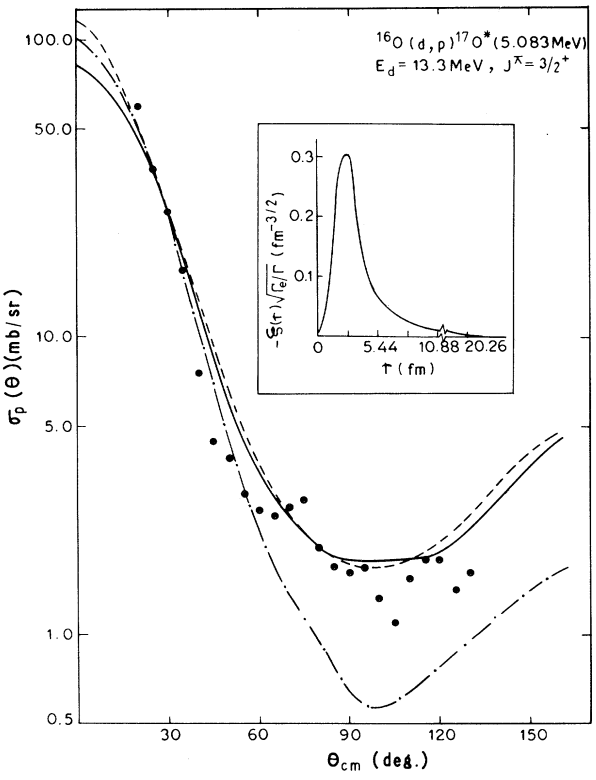

FIG. 1. DWBA calculation of the proton angular distribution for oxygen. The solid curve is with our wave packet for the square well, the dashed curve is by the damping method with the same square well, and the dash-dotted curve is by the damping method for the Woods-Saxon potential (Ref. 7). The solid circles are experimental data from Darden *et al.* (Ref. 9). Inset: $-(\Gamma_e/\Gamma)^{1/2}\xi(r)$ for ^{17}O*.

To demonstrate the viability of our model in applications we calculate $\sigma_p(\theta)$ for the reactions $^{16}O(d,p)^{17}O^*$ and $^{12}C(d,p)^{13}C^*$ using the DWBA. Since the Woods-Saxon potential is not analytically solvable, we have replaced the neutron-nucleus potential V_{nA} by a square well of range d and depth V_0 which corresponds to the given resonance energy E_α. This replacement is reasonable for low-energy resonances when shape dependence of the potential is not crucial. Now it is known that a given set of potential (whether Woods-Saxon or square-well) parameters reproducing the resonance momentum α correctly fails to yield the correct experimental width Γ_e. Further $\xi(r)$ behaves like $\Gamma^{1/2}\eta_l(\alpha r)$ outside the potential range, Γ being the single-particle width; therefore the natural way to make this behavior correspond to Γ_e is to scale $|\xi\rangle$ by a known factor[12] $(\Gamma_e/\Gamma)^{1/2}$. Hence we use $(\Gamma_e/\Gamma)^{1/2}|\xi_j\rangle$ for $|\chi_n\rangle$. The potential parameters used are $d=3.4$ fm, $V_0=36.7$ MeV, $\Gamma=71.6$ keV, and $\Gamma_e/\Gamma=1.26$ for $^{17}O^*$; and $d=22$ fm, $V_0=90.0$ MeV, $\Gamma=114.5$ keV, and $\Gamma_e/\Gamma=0.057$ for $^{13}C^*$.

The resonance wave function for $^{17}O^*$ is shown in the inset in Fig. 1; it falls rapidly with distance for $r>7$ fm. The DWBA integrations were indeed very stable after about 50 fm. We also see that Gaussian damping calculations with the same square well as ours give almost equally good fits to $\sigma_p(\theta)$ at almost all angles. However, whereas the normalization near forward angles is gauranteed in our case, a spectroscopic factor of 1.95 is needed for the latter. The damping-method fit using Woods-Saxon potential[7] becomes poor for $\theta > 60°$. The contour deformation method[9] as well as the damping method (with a spectroscopic factor of 0.07) result in poor fits for $\theta > 30°$ for carbon in Fig. 2 while our method gives rather satisfactory correspondence with experimental data. This is because our expansion coefficients contain a Breit-Wigner-*like* denominator multiplied by factors which take into account other structures in the momentum plane whereas in the resonance-peak method[9] the scattering wave function is arbitrarily fixed at the resonance momentum and the neutron energy is integrated out over a Breit-Wigner shape. Hence the features obtained by us are not only comparable but actually improved; this can be attributed to the sound principles on which the resonance wave function has been constructed. Let us mention that we have not attempted to fit the proton polarization data because of their sensitivity to the spin-orbit term rather than to the nature of $|\chi_n\rangle$.

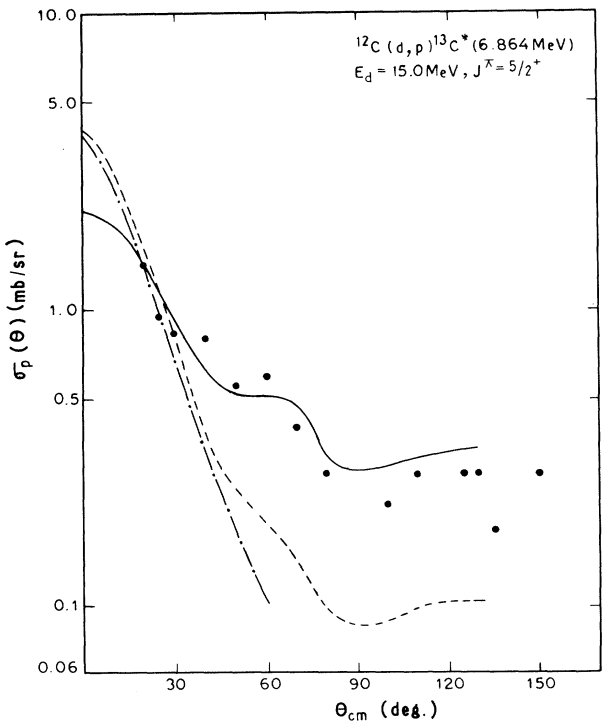

FIG. 2. As in Fig. 1 but for carbon. The dash-dotted curve is also from Darden *et al.* (Ref. 9).

We have also extended the theory to separable interactions and developed a scheme in which $|\xi_j\rangle \neq |\tilde{\xi}_j\rangle$. The form of ρ is so appealing that it should apply even to potentials with a tail provided $F(k)$ is chosen appropriately. Details shall be discussed elsewhere.[13]

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[12]Moreover this scaling also normalizes the vertex function $a(\alpha)$ from the old value $\sqrt{\Gamma}$ to the new value $\sqrt{\Gamma_e}$.
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