Fractal Dimension of Strange Attractors from Radius versus Size of Arbitrary Clusters

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A fractal dimension $D_{F'}$ of strange attractors is estimated as follows. Clusters of n nearest-neighbor points are sampled from a time series; $D_{F'}$ is found from $\bar{R}^{D_{F'}} \sim n$ where $\bar{R}$ is the average cluster's radius. The estimation of $D_{F'}$ is shown to be especially efficient for high dimensional systems.

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Dissipative chemical systems often exhibit a transition from regular to chaotic, i.e., erratic but still not entirely random, motion in phase space. These transitions are due to the presence of attractors in phase space, which are termed "strange." The fractal (or Hausdorff) dimension,[1] D_F, quantitatively measures this degree of "strangeness."[2,3] Classically, it is calculated as follows. Consider a d-dimensional phase space and let $M(\epsilon)$ be the number of d-dimensional boxes of side length ϵ required to cover a set of N points constituting the attractor. Then

$$M(\epsilon) \sim \epsilon^{-D_F}, \quad \epsilon \to 0, \ N \to \infty. \tag{1}$$

The determination of the attractor's volume becomes increasingly difficult as the dimension of the space increases.[2]

Alternative methods therefore have focused on the distribution of distances between the N points on the attractor. Consider the set $\{\vec{X}_i\}$, $i = 1, \ldots, N$, obtained from a time series of N consecutive measurements. Grassberger and Procaccia[4] have calculated a correlation exponent ν which is related to the number of pairs of points at distance $|\vec{X}_i - \vec{X}_j|$ less than ϵ. The computation of ν in multidimensional space is considerably less difficult since one deals with time series of N points, not with the volume. However, ν is related to the fractal dimension by $\nu \leq D_F$. Time series have been also used by Takens[5] to calculate rigorously the limit capacity of strange attractors, which is greater than or equal to the fractal dimension.

In this Letter we utilize an alternative definition of fractal dimension, $D_{F'}$, which shares the above convenient feature of dealing with the points, rather than with the volume. We choose arbitrary center points on the attractor and consider clusters of increasing size (i.e., number of points) enclosing these centers. $D_{F'}$ is then estimated from the radius vs size dependence of these clusters.

Let us return to the coverage of the attractor with boxes, but define boxes containing a constant number of points n, instead of a constant volume as usual. This, for example, can be realized as follows: The attractor is first divided into elongated boxes of base R^{d-1} and of height spanning the entire attractor. The height is subsequently divided to create boxes of variable height, each containing precisely n points. A difficulty arises at the perimeter of the attractor but this can be solved by discarding any box having a height much greater than, say, that of the preceding box. An analog of Eq. (1) for boxes of constant n is

$$M(n) \sim \bar{R}(n)^{-D_{F'}}, \quad n/N \to 0, \quad N \to \infty, \tag{1'}$$

where $\bar{R}(n)^d$ constitutes the average box volume.[6] Since $M(n)n = N = \text{const}$, Eq. (1') gives at once

$$n \sim \bar{R}(n)^{D_{F'}}. \tag{2}$$

Intuitively, we expect the two alternative definitions of a fractal dimension, i.e., D_F in Eq. (1) or $D_{F'}$ in Eqs. (1') and (2), to be equivalent. Admittedly, it may turn out after all that the two definitions are not strictly equivalent. In that case, however, there does not appear to be any *a priori* reason to prefer one over the other. Indeed Mandelbrot[7] employs an expression similar to Eq. (2) in order to discuss the fractal dimension of random walks. [There, $\bar{R}(n)$ is the (root mean square) average displacement for n steps.] As we shall see, from the point of view of computational ease, the definition employed here is to be preferred for strange attractors.

The average $\bar{R}(n)$ can be estimated with great ease with the help of a random sampling of clusters of size n. The sampling has to be elaborated. Very often the fractal is nonuniform, i.e., certain regions of the attractor have higher "seniority"[4] in the sense that they are visited more often than others. In that case, however, the number of boxes having n points each, within any region of the attractor, is proportional to the number of points inside that region. The correct

sampling of $R(n)$ is therefore automatically realized by picking points at random out of the total time series, calculating $R(n)$ of the spheres containing the $n-1$ nearest neighbors to these points, and averaging $R(n)$ over the sample. The sampling of $R(n)$ constitutes therefore a so-called "weight average"[8] over clusters.

Our algorithm for calculating D_F, from $\overline{R}(n)$ thus proceeds as follows. Consider a set $\{\vec{X}_i\}$, $i=1,\ldots,N$, of points on an attractor, obtained from a time series. A reference point $\vec{X}_j$ is chosen at random and all its (square) distances $(\vec{X}_i - \vec{X}_j)^2$ with the $N-1$ remaining points are computed. These distances are then sorted out in an ascending series $\{R^2(n,\vec{X}_j)\}$, $n=1,\ldots,N$, with a very efficient algorithm of the numerical algorithms group library, which determines $R^2(n,\vec{X}_j)$ as a function of n. The procedure is iterated over other reference points $\vec{X}_j$ and averages $\overline{R^2}(n)$ for a given n are taken.[9] A few hundred iterations, at most, suffice to observe the power law, Eq. (2), in the limit $\overline{R}(n) \to 0$, i.e., for $n \ll N$.

Our calculation of D_F, from Eq. (2), using the weight average $\overline{R^2}(n)$, has been tested for various finite- and infinite-dimensional systems. A typical calculation for a time series of $N=45\times10^3$ and a few hundred (say, 500) iterations for the estimation of $\overline{R^2}(n)$ took, depending on the model studied, between 20 and 40 min of central-processor-unit time on an IBM 370/165 and a relatively insignificant amount of memory. This is similar to the ease of estimating ν in Ref. 4 and should be contrasted with the huge difficulties associated with estimating D_F with box-counting algorithms.

Tests of our method were performed with three different two-dimensional maps: the one originally studied by Hénon[10] ($X_{i+1}=Y_i+1-aX_i^2$, $Y_{i+1}=bX_i$, with $a=1.4$ and $b=0.3$), the one introduced by Kaplan and Yorke[3] [$X_{i+1}=2X_i \pmod 1$, $Y_{i+1}=\alpha Y_i+\cos 4\pi X_i$ with $\alpha=0.2$], and the one studied by Zaslavskii[11] $\{X_{i+1}=[X_i+\nu(1+\mu Y_i)+\epsilon\nu\mu \times\cos 2\pi X_i] \pmod 1$, $Y_{i+1}=\exp(-\Gamma)(Y_i+\epsilon\cos 2\pi X_i)$, where $\mu=[1-\exp(-\Gamma)]/\Gamma$ with $\Gamma=3.0$, $\epsilon=0.3$, and $\nu=10^2\times\frac{4}{3}\}$. For the first two maps, we find excellent agreement with the power law of Eq. (2) (see Fig. 1), and the corresponding values for D_F, (see Table I) reproduce those obtained—with much more labor—by box-counting algorithms. The case of the Zaslavskii map seems to be exceptional. A calculation of ν for that system[4] did not show a clear-cut power law and an (admittedly poor) fit yielded $\nu \simeq 1.5$. A similar phenomenon seems to occur with box-counting algorithms and

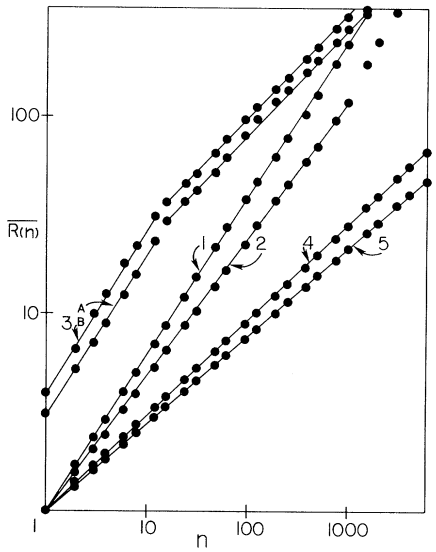

FIG. 1. Dependence of $\overline{R}(n)$ {or rather $[\overline{R^2}(n)]^{1/2}$; see Ref. 8} on n for various attractors. $\overline{R}(n)$ is the weight average (in arbitrary units) of the radius of a hypersphere that encloses $n+1$ points on the attractor. Line 1: Hénon map ($N=25\times10^3$); line 2: Kaplan-Yorke map ($N=25\times10^3$); line 3: Zaslavskii map (A: $N=65\times10^3$; B: $N=1\times10^6$); line 4: Lorenz equations ($N=25\times10^3$); line 5: Rabinovitch-Fabrikant equations ($N=65\times10^3$).

the value $D_F=1.38$ found in Ref. 2 is subject to caution.[4] (The latter value, moreover, does not satisfy the well-established bound[4] $\nu \leq D_F$.) A study of the Zaslavskii map with the help of our method clearly shows the presence of two power-law regimes, depending on the range of variation of n (see Fig. 1, curve 3A), giving the two values of D_F, in Table I. An increase of the resolution, obtained by taking $N=1\times10^6$, further confirms this conclusion (see curve 3B). (Incidentally, fitting the entire range by a single slope would lead to $D_F=1.67\pm0.27$.)

Two three-dimensional systems have been also considered: the Lorenz[15] model $\{dX/dt=\sigma(Y-X)$, $dY/dt=-Y-XZ+RX$, $dZ/dt=XY-bZ$ with $R=28$, $\sigma=10$, and $b=\frac{8}{3}\}$, and the Rabinovitch-Fabrikant[16] equations $\{dX/dt=Y(Z-1+X^2)+\gamma X$, $dY/dt=X(3Z+1-X^2)+\gamma Y$, $dZ/dt=2Z(\alpha+XY)$ with $\gamma=0.87$ and $\alpha=1.1\}$. In both cases, our results agree with the power law (Fig. 1, curves 4 and 5) and the corresponding values for D_F, reproduce those obtained—with much more labor—by box-counting algorithms.

For higher-dimensional systems, the use of box-counting algorithms is exceedingly difficult and this is where our approach becomes most

TABLE I. Correlation exponent ν and fractal dimensionalities D_F and $D_{F'}$ for various attractors. Results for $D_{F'}$ and their standard deviations have been obtained by taking ten values of the slope $\log[\bar{R}(n)]$ vs $\log(n)$ over the region $1 \leq n \leq N/10$. Values followed by an asterisk are those for the probabilistic dimension (Ref. 12), as obtained from the Kaplan-Yorke (Ref. 3) conjecture.

Model		ν	D_F [Eq. (1)]	$D_{F'}$ [Eq. (2)]
Hénon map		1.21 ± 0.01	1.26[a]	1.26 ± 0.01
Kaplan-Yorke map		1.42 ± 0.02	1.43[a]	1.43 ± 0.01
Zaslavskii map		≃ 1.5	1.38(?)[a]	1.22 ± 0.02; 1.80 ± 0.03
Lorenz equation		2.05 ± 0.01	[2.06 ± 0.01[b]]*	2.10 ± 0.04
Rabinovich-Fabrikant equation		2.19 ± 0.01	2.31[a]	2.29 ± 0.02
Mackey-Glass equations	$\tau = 17$	1.95 ± 0.03	2.13 ± 0.03[c]	2.10 ± 0.02 ($d = 10$)
	$\tau = 23$	2.42 ± 0.05	2.76 ± 0.06[c]	2.65 ± 0.03 ($d = 10$)
	$\tau = 30$	2.8 ± 0.3	[3.58 ± 0.04[c]]*	3.68 ± 0.06 ($d = 10$)
	$\tau = 100$	7.5 ± 0.2	[≃10[c]]*	12.6 ± 0.2 ($d = 30$)

[a]Ref. 2.
[b]Ref. 13.
[c]Ref. 14.

valuable. Thus, we have studied an "infinite"-dimensional system represented by the delay differential equation $\{dX(t)/dt = aX(t-\tau)/[1+X(t-\tau)^{10}] - bX(t)$ with $a = 0.2$ and $b = 0.1\}$, introduced by Mackey and Glass[17] as a model for the regeneration of blood cells in leukemia patients. For the purpose of the numerical investigation, $X(t)$ over the interval $[t, t-\tau]$ has been approximated by p samples taken at intervals $\Delta t = \tau/(p-1)$ with $p = 200$–500. The d-dimensional vectors in the time series were always chosen as $\vec{X}(t) = [X(t), X(t+\tau), X(t+2\tau), \ldots X(t+d\tau)]$ with $d \gg D_F$. Our results for the dependence of $\bar{R}(n)$ on n, for various values of τ, are presented in Fig. 2; $\tau = 17$ corresponds to a system only slightly above transition to chaos. All cases show a perfect power-law behavior, even for very large $\tau = 100$ at which the attractor has a highly complex structure. Our value $D_{F'} = 2.10 \pm 0.02$ for $\tau = 17$ (see Table I) agrees to within experimental accuracy with that obtained for D_F by Farmer.[14] At $\tau = 23$, a slight discrepancy appears between the values obtained with the two approaches, presumably because the box-counting algorithm of Ref. 14 uses an embedding dimension $d = 3$ which is only slightly higher than D_F. For $\tau = 30$ and $\tau = 100$, D_F becomes much larger than 3 and so does the minimal embedding dimension. For that reason, use of box-counting algorithms becomes impossible and the values followed by an asterisk in the third column of Table I are those for the probabilistic dimension,[12] as obtained in Ref. 14 using the Kaplan and Yorke conjecture.[3] Inspection of the table shows that, for these large values of τ, our $D_{F'}$ results are slightly higher than those for the probabilistic dimension. The ν values, on the other hand, are much lower and the discrepancy between ν and $D_{F'}$ is seen to increase with τ.

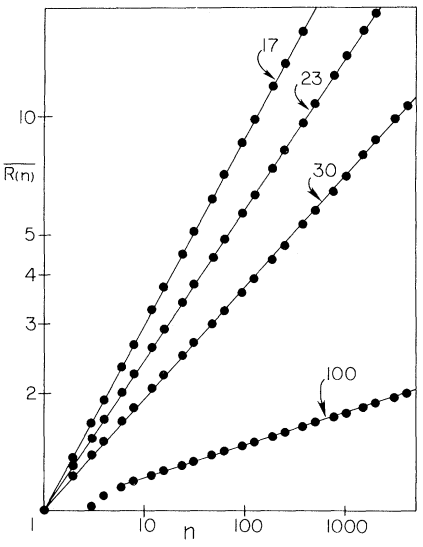

FIG. 2. Same as Fig. 1 for the Mackey-Glass equations. The lines are for $\tau = 17$ ($N = 65 \times 10^3$); $\tau = 23$ ($N = 45 \times 10^3$); $\tau = 30$ ($N = 45 \times 10^3$); and $\tau = 100$ ($N = 45 \times 10^3$). The embedding dimensions are $d = 10$ for $\tau = 17$, 23, and 30; $d = 30$ for $\tau = 100$.

To conclude, the numerical results presented above support the contention that a fractal dimension $D_{F'}$ of chaotic systems can be indeed calculated with great ease by use of Eq. (2) and an average $\bar{R}(n)$, estimated over a random sample of points. Our approach should be particularly useful in experimental situations which are typically of high dimension. The method might be also useful for the estimation of a fractal dimension for other collections of discrete objects.

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[6]Equation (1) requires implicitly that $\epsilon \to 0$ while $\bar{n} \gg 1$, where $\bar{n}$ is the average number of points per box. Similarly, Eq. (1′) requires that $n \gg 1$.

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