Competing Criticality of Short- and Infinite-Range Interactions on the Cayley Tree

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The Ising model, with equivalent-neighbor and nearest-neighbor interactions of Cayley-tree connectivity, is solved exactly. Breaking translational symmetry by turning on the Cayley interactions is analogous to lowering spatial dimensionality in Bravais lattices. A range of classical criticality, a point of logarithmic corrections, a range of continuously varying power-law singularities, and a point of exponential singularity are successively encountered.

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The study of competing short-range and long-range interactions is relevant to a variety of problems in statistical mechanics. This paper examines the Ising model on a Cayley tree subject to nearest-neighbor (short-range) and equivalent-neighbor (infinite-range) interactions. The model is solved exactly by the Hamiltonian minimization method,[1] and phase diagrams and critical exponents are obtained. The critical behavior is sensitive to a delicate balance of the competing influences of the two types of interactions. On Bravais lattices, any weak equivalent-neighbor interaction changes the nature of the transition from Ising to classical.[2,3] By contrast, in the Cayley tree, applying the infinite-range interaction results in a nonuniversal line of phase transitions. In this case increasing the strength of the nearest-neighbor interaction decreases the translational invariance of the spins, and is found to have effects similar to lowering of spatial dimensionality. This is reminiscent of dimensional reduction in a random magnetic field,[4] which is also a system with broken translational symmetry. We also speculate on nonclassical corrections to scaling above the upper critical dimensionality.

Cayley trees are hierarchical lattices,[5] quite generally characterized by the absence of translational symmetry.[6] For this reason Kaufman and Griffiths[6] have suggested that studies of models on hierarchical lattices can contribute to the understanding of low-symmetry systems such as surfaces and random magnets. It is important to note, however, that the translational symmetry is broken in a special manner, which can be called *hierarchical* as it is related to the generational structure of this type of lattice. This is different from the breaking of translational symmetry in Bravais lattices which is usually due to defects. The advantage of models on hierarchical lattices is that they are exactly solvable[5] and can provide insights into the behavior of more complex systems on Bravais lattices. For instance, frustrated hierarchical lattices have been employed to model spin-glasses,[7] and numerous models in solid-state physics[8] and statistical mechanics[9-16] have been studied on Cayley trees.

The competition between nearest-neighbor and equivalent-neighbor interactions on the Cayley tree can also be regarded as a competition between interactions that preserve translational invariance and interactions which break it. In the absence of nearest-neighbor coupling all spins are equivalent. This "translational invariance" is broken by the nearest-neighbor interactions that place spins in the hierarchical structure of the Cayley tree. The free energy of our model is related to that of the Ising model with nearest-neighbor interactions only, in a magnetic field.[12] Although the latter free energy becomes a nonanalytic function of the magnetic field below the Bethe-Peierls transition temperature, there is no spontaneous magnetization except at zero temperature. In the presence of equivalent-neighbor interactions, however, there is ordering at finite temperature. The disordering transition has a variety of theoretically interesting critical behaviors. As the strength of the nearest-neighbor coupling is increased, the nature of the phase transition changes from classical to behavior similar to a system at its upper critical dimensionality. Then, there is a nonclassical line, with exponents varying with the couplings, that terminates at a special point with a behavior similar to a system at its lower critical dimensionality. Thus, increasing the nearest-neighbor interaction (and thus reducing translational invariance), appears to have on the critical behavior effects similar to lowering the spatial dimensionality. Another system lacking translational symmetry for which dimensional reduction has been predicted[4] is a magnet in random fields.

We study a system of Ising spins $\sigma_i = \pm 1$, on the sites i of a Cayley tree of coordination number z, with both short-range and infinite-range interactions. The Hamiltonian is

$$\frac{-\mathcal{H}}{k_B T}=\frac{J}{2N}\sum_{i,j}\sigma_i\sigma_j+K\sum_{\langle ij\rangle}\sigma_i\sigma_j, \qquad (1)$$

where N is the total number of spins. The first summation involves all pairs of spins irrespective of their separation, while the second summation is over nearest neighbors only. We consider only the ferromagnetic case ($J, K\geq 0$).

For $K=0$, the Hamiltonian describes the equivalent-neighbor Ising model, which has a classical (mean-field) phase transition at $J=1$.[17] In this model all spins are equivalent and the system can be regarded as being translationally invariant. When K is nonzero, the spins are no longer equivalent as they are placed in the hierarchical structure of the Cayley tree. For $J=0$, Eq. (1) describes the Ising model on the Cayley tree with nearest-neighbor interactions only, which has been studied extensively.[10-16] The Bethe-Peierls approximation is exact for spins in the interior of the Cayley tree,[18] and thus a classical phase transition is predicted. However, the free energy at zero magnetic field is analytic at all temperatures.[10] This apparent contradiction is due to the unusual structure of the Cayley tree, in that a finite fraction of spins is located on the surface of the tree. The free energy in a magnetic field h, calculated by Müller-Hartmann and Zittartz,[12] is an analytic function of K and h at temperatures above the Bethe-Peierls transition temperature. Below this temperature the free energy is a nonanalytic function of the field h as h goes to zero. The leading nonanalytic behavior for $K>K_{\rm BP}=\tanh^{-1}[1/(z-1)]$ is $f_0(K,h)\sim|h|^{\Delta}$, where the exponent Δ varies with K as $\Delta(K)=\ln(z-1)/\ln[(z-1)\tanh K]$. This nonuniversal behavior can be understood by thinking of the Cayley tree as a hierarchical lattice with noniterated bonds.[6] When $\Delta(K)$ is an even integer, $\Delta(K_{2m})=2m$, with $m=1, 2, \ldots$, the free energy has a logarithmic singularity $f_0(K_{2m},h)\sim|h|^{2m}\ln(1/|h|)$.

The Hamiltonian in (1) with both J and K nonzero has been studied on Bravais lattices,[3] and the same Hamiltonian minimization method[1] can be applied to the Cayley tree. The partition function Z can be rewritten with use of Gaussian integrals as

$$Z=\left(\frac{NJ}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}dm\sum_{\{\sigma_i\}}\exp\left\{-\frac{NJm^2}{2}+Jm\sum_{i=1}^{N}\sigma_i+K\sum_{\langle ij\rangle}\sigma_i\sigma_j\right\}.$$

Ignoring terms of order $\ln N/N$ in the exponent gives

$$Z=e^{-Nf}\simeq\int_{-\infty}^{\infty}dm\exp\{-N[\tfrac{1}{2}Jm^2+f_0(K,Jm)]\}, \qquad (2)$$

where $f_0(K,Jm)$ is the free energy per spin of the Cayley tree with nearest-neighbor interactions K, and subject to a magnetic field $h=Jm$. In the thermodynamic limit $N\to\infty$, applying the saddle-point method to Eq. (2) gives the free energy f as

$$f(K,J)=\min[\tfrac{1}{2}Jm^2+f_0(K,Jm)]_m.$$

The value of m minimizing this expression is the net magnetization of the system,[1] and will be denoted $\bar{m}$.

For small enough values of the couplings J and K, the system is disordered and $\psi(m)=\frac{1}{2}Jm^2+f_0(K,Jm)$ is minimized for $\bar{m}=0$. For large values of couplings, the system develops a spontaneous magnetization. This is indicated by $\psi(m)$ achieving its absolute minumum for $\bar{m}\neq 0$. For continuous transitions, the phase boundary separating the disordered and ordered states is obtained by the usual requirement that $\psi(m)$ should be marginally convex at $m=0$, i.e.,

$$d^2\psi(m)/dm^2|_{m=0}=J-J^2\chi_2(K)=0,$$

where $\chi_2=-\partial^2 f_0/\partial h^2|_{h=0}$ is the zero-field susceptibility of the short-range model.[11] The critical boundary [Fig. 1(a)] is given by

$$J=\begin{cases}\dfrac{1-(z-1)\tanh^2K}{(1+\tanh K)^2}, & K\leq K_2,\\ 0, & K\geq K_2,\end{cases} \qquad (3)$$

where $\tanh K_2=1/(z-1)^{1/2}$. Close to the critical line the magnetization $\bar{m}$ is small and it is sufficient to approximate $\psi(m)$ by an expansion in powers of m. The critical line can be divided into the following segments:

(i) For $K<K_{\rm BP}=\tanh^{-1}[1/(z-1)]$, the free energy f_0 is an analytic function of h, and the expansion of $\psi(m)$ will also be analytic. The phase transitions in this case are classical with $\beta=\frac{1}{2}$ and $\alpha=0$ (discontinuity).

(ii) For $K_{\rm BP}<K<K_4=\tanh^{-1}[1/(z-1)^{3/4}]$, f_0 and hence $\psi(m)$ are nonanalytic ($\Delta>4$). However, the leading nonanalytic behavior in $\psi(m)$ occurs for a power larger than 4 and $\psi(m)$ can be expanded as

$$\psi(m)\simeq f_0(K,0)+\frac{J\chi_2}{2}\left(\frac{1}{\chi_2}-J\right)m^2-\frac{1}{4!}\chi_4J^4m^4,$$

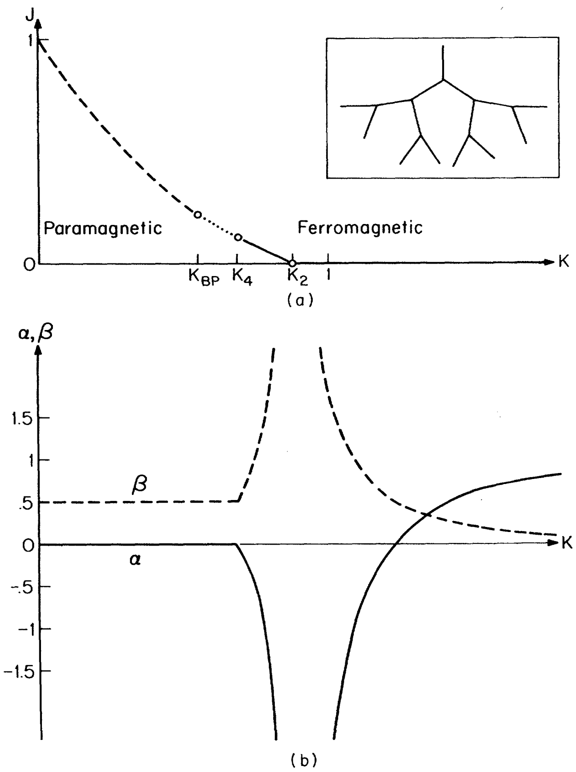

FIG. 1. (a) Phase diagram with equivalent- and nearest-neighbor interactions J,K on the Cayley tree with $z=3$ (inset). The critical line is composed of segments of classical criticality (dashed line), classical leading singularities and nonclassical corrections to scaling (dotted line), and continuously varying exponents (solid line). (b) Exponents α (solid line) and β (dashed line) vs the coupling K.

where $\chi_4 = -\partial^4 f_0/\partial h^4|_{h=0}$, the fourth-order susceptibility, is a negative quantity[14,16] which insures the existence of finite minima. Minimizing $\psi(m)$ still leads to classical exponents, but there will be nonclassical singularities in higher-order derivatives of the free energy.

(iii) As $K \to K_4$ the fourth-order susceptibility χ_4 diverges[12] as $(K_4 - K)^{-1}$. At $K = K_4$, the expansion for $\psi(m)$ is

$$\psi(m) \simeq f_0(K, 0) + \frac{J\chi_2}{2}\left(\frac{1}{\chi_2} - J\right) m^2 - b_4(Jm)^4 \ln\left(\frac{1}{J|m|}\right),$$

where b_4 is a negative constant.[14,16] For $t = J - 1/\chi_2$ negative the magnetization $\bar{m}$ is zero, while for positive t it behaves as $\bar{m} \sim |t/\ln t|^{1/2}$. The thermal response function is no longer discontinuous, but has a singularity: $\partial^2 f/\partial J^2 \sim 1/\ln(t)$. This critical behavior is reminiscent of a system at its upper critical dimensionality,[19] where the classical critical behavior ($\alpha = 0$, $\beta = \frac{1}{2}$) is modified by logarithmic corrections.

(iv) For $K_4 < K < K_2$ the singularity in f_0 is strong enough ($2 < \Delta < 4$) to modify the critical exponents completely. The expansion is now

$$\psi(m) \simeq f_0(K, 0) + \tfrac{1}{2}J\chi_2(\chi_2^{-1} - J)m^2 - AJ^\Delta m^\Delta,$$

with A a negative quantity.[13] Minimizing $\psi(m)$ gives the critical exponents in $\partial^2 f/\partial J^2 \sim t^{-\alpha}$ and $\bar{m} \sim t^{-\beta}$ as $\alpha = -(4-\Delta)/(\Delta - 2)$ and $\beta = (\Delta - 2)^{-1}$. Therefore, this segment is a nonuniversal line of critical points with continuously varying exponents ($0 > \alpha > -\infty$, $\frac{1}{2} < \beta < \infty$).

(v) As $K \to K_2$, the susceptibility χ_2 diverges, and for $K = K_2$ the appropriate expansion for $\psi(m)$ is

$$\psi(m) \simeq f_0(K, 0) - b_2 J^2 m^2 \ln[(J|m|)^{-1}] + \tfrac{1}{2}Jm^2,$$

where

$$b_2 = \frac{[(z-1)^{1/2}+1]^2}{(z-1)\ln(z-1)} > 0.$$

As $J \to 0$ the magnetization $\bar{m}$ and $\partial^2 f/\partial J^2$ behave as $\bar{m} \sim \exp[-(2b_2 J)^{-1}]$ and $\partial^2 f/\partial J^2 \sim \exp[-(b_2 J)^{-1}]$. This exponential singularity of thermodynamic quantities is typical of systems at their lower critical dimensionality, such as the one-dimensional Ising model[20] or the two-dimensional XY model.[21]

(vi) For $K > K_2$, Δ is smaller than 2, and $\psi(m) \simeq f_0(K, 0) - AJ^\Delta m^\Delta + \frac{1}{2}Jm^2$. The system is magnetized; however, as $J \to 0$ the magnetization goes to zero as J^β, while $\partial^2 f/\partial J^2 \sim J^{-\alpha}$. The exponents α and β are now given by $\alpha = (4 - 3\Delta)/(2 - \Delta)$ and $\beta = (\Delta - 1)/(2 - \Delta)$. For large enough K ($\Delta < \frac{4}{3}$), α becomes positive indicating a divergence in $\partial^2 f/\partial J^2$. The amplitude A is equal to a constant plus a numerically small periodic function of $\ln J|m|$,[13] which slightly modifies the power-law behaviors of $\bar{m}$ and $\partial^2 f/\partial J^2$. The exponents α and β [Fig. 1(b)] are related by $\alpha + 2\beta = 1$, implying that γ retains its classical value of unity.

We observe that as the coupling K is increased from zero the nature of the phase transition initially remains classical until a critical value ($K = K_4$) where logarithmic corrections appear similar to behavior at an upper critical dimensionality. As K is further increased the critical behavior is nonclassical power law, until another critical value ($K = K_2$) where the exponential singularities resemble those of a system at its lower critical dimensionality. Thus increasing the coupling K (and breaking the equivalence between the spins) appears to be the same as lowering the dimensionality of the system. On the basis of this analogy, we can also speculate that above the upper critical dimension the leading classical behavior is accompanied by nonclassical singularities of higher-order response functions, similar to $K_{\rm BP} < K < K_4$. It is interesting that dimensional reduction is predicted in spin systems with random magnetic fields which are also characterized by broken translational invariance. Potts and percolation models on Cayley trees with infinite-range interactions exhibit dimensional reduction and will be discussed in forthcoming papers.

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